Application of WKB theory for the simulation of the weakly nonlinear dynamics of gravity waves

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Motivation: GW Impacts

Gravity-wave effects numerous, e.g.
- Clear-air turbulence (e.g. Koch et al 2005)
- Clouds & convection (e.g. Zhang et al 2001, 2003)
- Middle-atmosphere waves (QBO, solar tides)
- residual circulation
  - GW impact in stratosphere (e.g. Palmer et al 1986)
  - GW control in mesosphere (e.g. Lindzen 1981)
- Indirectly: Impact middle atmosphere on troposphere (downward control)


Scaife et al (2005)
Motivation: Parameterization of GW Processes

GW propagation

- Described using WKB theory (Grimshaw 1975, …)
- **Simplifications** for efficiency:
  - Single-column
  - Steady state
  - Transience considered important (intermittency, Alexander et al 2010)
  - Horizontal propagation has an effect (Dunkerton1984, …, Kawatani et al 2010)
- **Synoptic-scale balanced background assumed**
  - But NWP models resolve some GWs
  - Theory to be revisited

Effects on GW-ST IA (Senf & Achatz 2011)
Motivation: Parameterization of GW Processes

GW propagation
• Described using WKB theory (Grimshaw 1975, …)
• **Simplifications** for efficiency:
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  • Steady state
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  • But NWP models resolve some GWs
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Nonlinear dissipation
• accelerated by wave-mean flow interaction (Dosser & Sutherland 2011)

Rieper, Achatz & Klein (2013)
Objectives & Strategy

**Goal:**

- A **prognostic WKB model** for propagation and dissipation of subgrid-scale GWs
- to be implemented into NWP and climate models.

**Strategy:**

- **WKB theory**
- **Validation against LES**
- **Numerical methods**
Preliminary Work

LES code pseudo-incompressible equations (Rieper, Hickel & Achatz 2013)
4D ray tracing (Senf & Achatz 2011)
- applied to interaction GWs with solar tides
- single column and steady state lead to overestimation of GW impact
- refraction by horizontal gradients leads to considerable latitudinal displacements

true
no transience

no hor. refraction

u and c (m/s)
suppression of critical layers
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- Scaling of 2D Euler so that waves are close to convective instability
- Scale separation parameter $\epsilon = L/H_{\theta}$

$$
\mathbf{\hat{v}} = \mathbf{\tilde{v}}^{(0)}
$$

$$
\mathbf{\hat{\theta}} = \mathbf{\tilde{\theta}}^{(0)} + \epsilon \mathbf{\tilde{\theta}}^{(1)}
$$

$$
\mathbf{\hat{\pi}} = \mathbf{\tilde{\pi}}^{(0)} + \epsilon^2 \mathbf{\tilde{\pi}}^{(2)}
$$

$$
\mathbf{\tilde{v}}^{(0)} = \mathbf{\hat{V}}^{(0)} + \epsilon \mathbf{\hat{V}}^{(1)} + o(\epsilon) \quad \text{(e.g.)}
$$
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- Scaling of 2D Euler so that waves are close to convective instability
- Scale separation parameter $\epsilon = L/H_\theta$

$$\hat{\mathbf{v}} = \tilde{\mathbf{v}}^{(0)}$$
$$\hat{\theta} = \tilde{\theta}^{(0)} + \epsilon \tilde{\theta}^{(1)}$$
$$\hat{\pi} = \tilde{\pi}^{(0)} + \epsilon^2 \tilde{\pi}^{(2)}$$

$$\tilde{\mathbf{v}}^{(0)} = \hat{\mathbf{V}}^{(0)} + \epsilon \hat{\mathbf{V}}^{(1)} + O(\epsilon) \quad \text{(e.g.)}$$

$$\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\xi_\tau, \xi_\chi, \xi_\zeta) + \Re \left\{ \hat{\mathbf{V}}_1^{(0)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\epsilon} \varphi(\tau, \chi, \zeta) \right] \right\}$$

Mean flow with only large-scale dependence
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- Scaling of 2D Euler so that waves are close to convective instability
- Scale separation parameter $\epsilon = L/H_\theta$

\[
\hat{\mathbf{V}} = \tilde{\mathbf{V}}^{(0)}
\]
\[
\hat{\vartheta} = \hat{\vartheta}^{(0)} + \epsilon \hat{\vartheta}^{(1)}
\]
\[
\hat{\pi} = \hat{\pi}^{(0)} + \epsilon^2 \hat{\pi}^{(2)}
\]

\[
\tilde{\mathbf{v}}^{(0)} = \hat{\mathbf{V}}^{(0)} + \epsilon \hat{\mathbf{V}}^{(1)} + o(\epsilon) \quad \text{(e.g.)}
\]
\[
\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\hat{\vartheta}, \hat{\vartheta}, \hat{\pi}) + \Re \left\{ \hat{\mathbf{V}}_1^{(0)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\epsilon} \varphi(\tau, \chi, \zeta) \right] \right\}
\]

Wavepacket with amplitude, wavenumber and frequency with large-scale dependence

\[
k = \nabla_{(\chi, \zeta)} \varphi
\]
\[
\omega = -\frac{\partial \varphi}{\partial \tau}
\]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- Scaling of 2D Euler so that waves are close to convective instability
- Scale separation parameter $\epsilon = L/H_\theta$

\[
\hat{v} = \tilde{V}^{(0)} \\
\hat{\theta} = \tilde{\theta}^{(0)} + \epsilon \tilde{\theta}^{(1)} \\
\hat{\pi} = \tilde{\pi}^{(0)} + \epsilon^2 \tilde{\pi}^{(2)}
\]

\[
\tilde{V}^{(0)} = \hat{V}^{(0)} + \epsilon \hat{V}^{(1)} + o(\epsilon) \quad (\text{e.g.})
\]

\[
\hat{V}^{(0)} = \hat{V}_0^{(0)}(\xi, \chi, \zeta) + \Re \left\{ \hat{V}_1^{(0)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\epsilon} \varphi(\tau, \chi, \zeta) \right] \right\}
\]

\[
\hat{V}^{(1)} = \hat{V}_0^{(1)}(\tau, \chi, \zeta) + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{V}_\alpha^{(1)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\epsilon} \alpha \varphi(\tau, \chi, \zeta) \right] \right\}
\]

Next-order mean flow
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- Scaling of 2D Euler so that waves are close to convective instability
- Scale separation parameter $\epsilon = L/H_\theta$

$$\hat{v} = \tilde{v}^{(0)}$$
$$\hat{\theta} = \tilde{\theta}^{(0)} + \epsilon \tilde{\theta}^{(1)}$$
$$\hat{\pi} = \tilde{\pi}^{(0)} + \epsilon^2 \tilde{\pi}^{(2)}$$

$$\tilde{v}^{(0)} = \hat{V}^{(0)} + \epsilon \hat{V}^{(1)} + o(\epsilon) \quad (\text{e.g.})$$

$$\hat{V}^{(0)} = \hat{V}_0^{(0)}(\xi, \eta, \xi) + \mathcal{R} \sum_{\alpha=1}^{\infty} \hat{V}_\alpha^{(1)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\epsilon} \alpha \varphi(\tau, \chi, \zeta) \right]$$

Harmonics of the wavepacket due to nonlinear interactions
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

Leading (zeroeth) order:

\[ i k \cdot \hat{V}_1^{(0)} = 0 \]

\[
\begin{pmatrix}
- i \hat{\omega} & 0 & 0 & ik \\
0 & - i \hat{\omega} & - N & im \\
0 & N & - i \hat{\omega} & 0 \\
1k & im & 0 & 0 \\
\end{pmatrix}
M(\hat{\omega}, k)
\]

\[ \hat{\omega} = \omega - k \hat{U}_0^{(0)} \]

intrinsic frequency
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

Leading (zeroeth) order: *dispersion relation and structure as from Boussinesq*

\[
i k \cdot \hat{V}_1^{(0)} = 0
\]

\[
\begin{pmatrix}
-i\hat{\omega} & 0 & 0 & ik \\
0 & -i\hat{\omega} & -N & im \\
0 & N & -i\hat{\omega} & 0 \\
iki & im & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\hat{U}_1^{(0)} \\
\hat{W}_1^{(0)} \\
1 \hat{\Theta}_1^{(1)} \\
N \hat{\Theta}^{(0)} \hat{\Pi}_1^{(2)}
\end{pmatrix}
= 0
\]

\[
M(\hat{\omega}, k)
\]

\[
\hat{\omega} = \omega - k\hat{U}_0^{(0)} \quad \text{intrinsic frequency}
\]

\[
\det(M) = 0 \Rightarrow
\]

\[
\hat{\omega}^2 = N^2 \frac{k^2}{k^2 + m^2}
\]

\[
\begin{pmatrix}
\hat{U}_1^{(0)} \\
\hat{W}_1^{(0)} \\
1 \hat{\Theta}_1^{(1)} \\
N \hat{\Theta}^{(0)} \hat{\Pi}_1^{(2)}
\end{pmatrix}
= \text{Nullvector of } M
\]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order:

\[ M(\hat{\omega}, k) \begin{pmatrix} \hat{U}_1^{(1)} \\ \hat{W}_1^{(1)} \\ 1 \hat{\Theta}_1^{(2)} \\ \frac{N \hat{\theta}^{(0)}}{\hat{\theta}^{(0)} \hat{\Pi}_1^{(3)}} \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \end{pmatrix} \]

\[ = \begin{pmatrix} \frac{\partial \hat{U}_1^{(0)}}{\partial \chi} - \frac{\partial \hat{W}_1^{(0)}}{\partial \zeta} & - 1 - \kappa \frac{\hat{W}_1^{(0)}}{\kappa} & \frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} \\ \cdots & \cdots & \cdots \end{pmatrix} \]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order: Solvability condition leads to wave-action conservation

\[
M(\hat{\omega}, k) \begin{pmatrix}
\hat{U}_1^{(1)} \\
\hat{W}_1^{(1)} \\
1 \hat{\Theta}_1^{(2)} \\
N \hat{\theta}^{(0)} \\
\hat{\theta}^{(0)} \hat{\Pi}_1^{(3)}
\end{pmatrix} = \begin{pmatrix}
\ldots \\
\ldots \\
\ldots \\
- \partial \hat{U}_1^{(0)}/\partial \chi - \partial \hat{W}_1^{(0)}/\partial \zeta - 1 - \kappa \hat{W}_1^{(0)}/\partial \zeta \\
\kappa \hat{\pi}^{(0)}/\partial \zeta
\end{pmatrix}
\]

\[
\frac{\partial}{\partial \tau} \left( \frac{E'}{\hat{\omega}} \right) + \nabla_{(\chi, \zeta)} \cdot \left( c_g \frac{E'}{\hat{\omega}} \right) = 0
\]

\[
E' = \frac{\hat{\rho}^{(0)}}{2} \left( \frac{\hat{V}_1^{(0)}}{2} \right)^2 + \frac{1}{2N^2} \left( \frac{\hat{\Theta}_1^{(0)}}{\hat{\theta}^{(0)}} \right)^2
\]

\[
c_g = \left( \hat{U}_0^{(0)} + \frac{\partial \hat{\omega}}{\partial k}, \frac{\partial \hat{\omega}}{\partial m} \right)
\]

wave energy

group velocity
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order:

\[ \hat{V}^{(1)} = \ldots + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{V}_{\alpha}^{(1)}(\tau, \chi, \zeta) \exp \left[ \frac{i}{\varepsilon} \alpha \phi(\tau, \chi, \zeta) \right] \right\} \]

\[ \alpha = 2: \]

\[
M(2\hat{\omega}, 2k) \begin{pmatrix} \hat{U}_2^{(1)} \\ \hat{W}_2^{(1)} \\ \hat{\Theta}_2^{(2)} \\ N \hat{\Theta}^{(0)} \\ \hat{\Theta}^{(0)} \hat{\Pi}_2^{(3)} \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \end{pmatrix} \Rightarrow
\]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order: 2nd harmonics are slaved

\[ \hat{V}^{(1)} = \ldots + R \sum_{\alpha=1}^{\infty} \{ \hat{V}_{\alpha}^{(1)} (\tau, \chi, \zeta) \exp \left[ \frac{i}{\varepsilon} \alpha \varphi (\tau, \chi, \zeta) \right] \} \]

\[ \alpha = 2: \]

\[
M(2\hat{\omega}, 2k) \begin{pmatrix}
\hat{U}^{(1)}_2 \\
\hat{W}^{(1)}_2 \\
\frac{1}{N} \hat{\Theta}^{(2)}_2 \\
\frac{\hat{\Theta}^{(0)}_2}{\hat{\Theta}^{(0)}_2 + \pi_2^{(3)}}
\end{pmatrix} = \begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{pmatrix} \Rightarrow
\]

\[
M^{-1}(2\hat{\omega}, 2k) \begin{pmatrix}
\hat{U}^{(1)}_2 \\
\hat{W}^{(1)}_2 \\
\frac{1}{N} \hat{\Theta}^{(2)}_2 \\
\frac{\hat{\Theta}^{(0)}_2}{\hat{\Theta}^{(0)}_2 + \pi_2^{(3)}}
\end{pmatrix} = \begin{pmatrix}
\cdots \\
\cdots \\
\cdots \\
\cdots
\end{pmatrix}
\]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order: mean-flow acceleration by GW momentum-flux divergence

\[
\frac{\partial \hat{U}^{(0)}}{\partial \tau} + \cdots = -\nabla \cdot \hat{F}_{GW}^U
\]
Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

Validation (Rieper, Achatz & Klein 2013)

2nd harmonic

LES

WKB
Large-Amplitude WKB  
(Achatz, Klein & Senf 2010)

Validation (Rieper, Achatz & Klein 2013)

Problems as soon as rays tend to cross (caustics) and GW affects the mean flow
Caustics in WKB

Summary WKB (here Boussinesq, large-scale spatial dependence 1D)

• e.g. buoyancy

\[ \hat{B} = \Re \left\{ b(\zeta, \tau) e^{i [k x + \phi(\zeta, \tau)/\epsilon]} \right\} \]

• unique local wavenumber and frequency

\[
m(\zeta, \tau) = \frac{\partial \phi}{\partial \zeta}, \quad \omega = -\frac{\partial \phi}{\partial \tau} = kU(\zeta, \tau) \pm N(\zeta, \tau) \sqrt{\frac{k^2}{k^2 + m^2}} = \Omega(\zeta, \tau, k, m)
\]

• all other wave fields \( u, w \) from \( b, m \) and \( \omega \) by polarization relations

• along rays

\[
\frac{D_g m}{D\tau} = \frac{\partial m}{\partial \tau} + c_g \frac{\partial m}{\partial \zeta} = -\frac{\partial \Omega}{\partial \zeta}, \quad \frac{D_g \zeta}{D\tau} = c_g = \frac{\partial \Omega}{\partial m}
\]

• amplitude from wave action \( A = E'/\hat{\omega} \)

\[
\frac{D_g A}{D\tau} = \frac{\partial A}{\partial t} + c_g \frac{\partial A}{\partial \zeta} = -\frac{\partial c_g}{\partial \zeta} A + D, \quad c_g = \frac{\partial \omega}{\partial m}
\]

• mean wind

\[
\frac{\partial U}{\partial \tau} = -\frac{\partial}{\partial \zeta} (\bar{u} w), \quad \bar{u} w = A f(k, m, N)
\]
Caustics in WKB

Nonuniqueness of wave number arises easily:

e.g. reflection at a jet
Caustics in WKB

Nonuniqueness of wave number arises easily:

e.g. overtaking rays
Caustics in WKB

Nonuniqueness of wave number arises easily:

e.g. by wave-induced mean flow
Phase-Space Wave-Action Density
(Muraschko et al 2014)

- **linear limit:**
  wave field can be decomposed into fields with singlevalued wavenumbers

\[
\hat{B} = \Re \left\{ b_1(\zeta, \tau) \ e^{i [kx + \phi_1(\zeta, \tau)/\epsilon]} + b_2(\zeta, \tau) \ e^{i [kx + \phi_2(\zeta, \tau)/\epsilon]} \right\}
\]

\[
\frac{\partial \phi_1}{\partial \zeta} = m_1 \quad \frac{\partial \phi_2}{\partial \zeta} = m_2
\]

\[
\frac{D g^\alpha A_\alpha}{D \tau} = \frac{\partial A_\alpha}{\partial \tau} + c_{g^\alpha} \frac{\partial A_\alpha}{\partial \zeta} = -\frac{\partial c_{g^\alpha}}{\partial \zeta} A_\alpha + D_\alpha \quad (\alpha = 1, 2)
\]

Case dependent surgery: **very complex**
Phase-Space Wave-Action Density
(Muraschko et al 2014)

• **linear limit:**
  wave field can be decomposed into fields with singlevalued wavenumbers

• **Switch to phase space** does this automatically

phase-space wave-action density:

\[ \mathcal{N}(m, \zeta, \tau) = \int d\alpha A_{\alpha}(\zeta, \tau) \delta[m - m_{\alpha}(\zeta, \tau)] \]

conservation equation (for \( D = 0 \))

\[ \frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta} \left( c_g \mathcal{N} \right) + \frac{\partial}{\partial m} (m \mathcal{N}) = 0 \]

mean flow by

\[ \frac{\partial U}{\partial \tau} = -\frac{\partial}{\partial \zeta} (\overline{uw}) \quad \overline{uw} = \int dm \mathcal{N} f(k, m, N) \]

*generalization to 3D trivial*
Phase-Space Wave-Action Density
(Muraschko et al 2014)

1st numerical method: Eulerian model
• solve conservation equation

\[
\frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta} \left( c_g \, \mathcal{N} \right) + \frac{\partial}{\partial m} \left( m \, \mathcal{N} \right) = 0
\]

on grid in phase space using finite volume scheme (MUSCL)

• Momentum equation

\[
\frac{\partial U}{\partial \tau} = - \frac{\partial}{\partial \zeta} (uw) \quad uw = \int dm \, \mathcal{N} \, f(k, m, N)
\]

by finite difference

Would be too expensive in 6D!
Phase-Space Wave-Action Density
(Muraschko et al 2014)

2nd numerical method: Lagrangian model (ray tracer)
- Phase-space velocity is non-divergent

\[
\frac{\partial c_g}{\partial \zeta} + \frac{\partial \dot{m}}{\partial m} = \frac{\partial}{\partial \zeta} \frac{\partial \Omega}{\partial \dot{m}} + \frac{\partial}{\partial m} \left( - \frac{\partial \Omega}{\partial \zeta} \right) = 0
\]

- Hence flow is volume preserving
- Hence phase-space wave-action density conserved on rays

\[
\frac{D_g N}{D\tau} = \frac{\partial N}{\partial \tau} + c_g \frac{\partial N}{\partial \zeta} + \dot{m} \frac{\partial N}{\partial m} = 0
\]

- Region of nonzero \(N\) approximated by rectangles
- Rectangles move with central ray
- Rectangles change height (\(\Delta \zeta\)) and width (\(\Delta m\)) in area-preserving manner

Very efficient!
Simple test case: **Gaussian wave packet** \((a_0 = \text{amplitude wrt static instability})\)

\[
\begin{align*}
  b'(x, z, t = 0) &= A_b(z) \cos(kx + m_0 z) \\
  u'(x, z, t = 0) &= A_b(z) \frac{m_0 \hat{\omega}_0}{k N_0^2} \sin(kx + m_0 z) \\
  w'(x, z, t = 0) &= -A_b(z) \frac{\hat{\omega}_0}{N_0^2} \sin(kx + m_0 z)
\end{align*}
\]

\[
A_b(z) = a_0 \frac{N_0^2}{m_0} e^{-\frac{(z-z_0)^2}{2\sigma^2}}
\]

**WKB initialized with**

\[
N(m, z, t = 0) = \begin{cases} 
  \frac{A_b^2(z)}{2N_0^2 \hat{\omega}_0} & \text{for } m_0 - \frac{1}{2} \Delta m_0 < m < m_0 + \frac{1}{2} \Delta m_0 \\
  0 & \text{otherwise}
\end{cases}
\]
Phase-Space Wave-Action Density
(Muraschko et al 2014)

Hydrostatic wave packet
Phase-Space Wave-Action Density
(Muraschko et al 2014)

Hydrostatic wave packet:
Wave energy and induced mean flow at $t = 200\text{min}$
Phase-Space Wave-Action Density
(Muraschko et al 2014)

Wave packet reflected by a jet
Phase-Space Wave-Action Density (Muraschko 2014)

Non-Boussinesq: Wave packet refracted by jet
Phase-Space Wave-Action Density
(Muraschko 2014)

Non-Boussinesq: Wave packet reflected by a jet
Summary

- **Application of WKB to two-way interaction between propagating GW packet and induced mean flow**

- **Phase-space wave-action density** helps avoiding numerical instabilities due to caustics

- **Lagrangian approach (ray tracer)** numerically efficient
Outlook: DFG research unit MS-Gwaves

• Investigation **multi-scale dynamics of GWs** in 6 projects
• **prognostic WKB GW parameterization** to be developed for NWP and climate model
• **To be addressed:**
  • Sources
  • Propagation
  • dissipation
• **Combined effort:**
  • Theory,
  • modelling,
  • measurements,
  • laboratory experiments