Variational balance relations and applications
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Plan
1. Variational balance relations
2. Balance models in fluid dynamics
3. PDE case study: the semilinear Klein–Gordon equation
1. Why balance models?

**Balance relation as gravity wave diagnostics**
- High-order balance relations?
- Mathematical properties?
- Numerical implementation?
- Data assimilation

**Balance models as limiting test case for full models**
- Fast rotating limits cause scale separation!

**General method for certain singular perturbation problems?**
- Systems with strong gyroscopic forces
- Non-relativistic limit of semilinear Klein–Gordon
- Modified equations for variational time integrators?
1.1. Why variational?

Rigid construction
- Understand conservation law structure
- Noether’s theorem persists under model reduction
- For fluids: get conservation of energy and balance model PV

Flexible construction
- Variational balance relations are far from unique
- Use this freedom to get well-posedness in standard setting
- In examples: easy choice is often a good choice
1.2. Idea

Famility of Lagrangians with small parameter $\varepsilon$:

$$0 = \delta S = \delta \int_{t_1}^{t_2} L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) \, dt = \int_{t_1}^{t_2} \delta q_\varepsilon^T \left( D_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) - \frac{d}{dt} D_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) \right) \, dt$$

so that

$$E L_\varepsilon[q_\varepsilon] \equiv D_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) - \frac{d}{dt} D_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) = 0$$

Introduce transformation $q_\varepsilon = \Phi[q]$:

$$0 = \delta S = \int_{t_1}^{t_2} \delta q^T D\Phi[q]^* \left( D_q L_\varepsilon(\Phi[q], \frac{d}{dt} \Phi[q]) - \frac{d}{dt} D_q L_\varepsilon(\Phi[q], \frac{d}{dt} \Phi[q]) \right) \, dt$$

So Euler–Lagrange equation reads

$$D\Phi[q]^* \ E L_\varepsilon[\Phi[q]] = 0$$

Now choose $\Phi$ such that

$$D\Phi[q]^* \ E L_\varepsilon[\Phi[q]] = E L_{\text{slow}}^n[q] + O(\varepsilon^{n+1})$$
1.3. Turning the construction into a proof

From before:

\[ D\Phi[q] \ast EL_\varepsilon[\Phi[q]] = EL_{\text{slow}}[q] + \varepsilon^{n+1} EL_R^n[q] \]

Take a solution \( q \) of the slow equation:

- \( EL_{\text{slow}}^n[q] = 0 \) by definition
- Any derivative of \( q \) is \( O(1) \)
- Consequently, \( EL_R^n[q] = O(1) \)
- Then \( EL_\varepsilon[\Phi[q]] = O(\varepsilon^{n+1}) \)

Conclusion:

\( z \equiv \Phi[q] \) satisfies the full equation up to an \( O(\varepsilon^{n+1}) \) remainder.

Now use non-variational stability estimates to control the difference \( q_\varepsilon - z \)
2. Lagrangian fluid dynamics

For fluids, the configuration space is the group of flow maps $\eta$.

- Lagrangian vs. Eulerian fluid velocity: $\dot{\eta} = u \circ \eta$
- Lagrangian vs. Eulerian variation: $\delta \eta = w \circ \eta$
- Lagrangian vs. Eulerian transformation: $\eta' = v \circ \eta$

Note: Affine Lagrangians (Lagrangians which are linear in the velocity) lead to kinematic Euler–Lagrange equations in Eulerian variables!
2.1. Example: Rotating shallow water

\[
\varepsilon (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) + f \mathbf{u}^\perp + \frac{B_u}{\varepsilon} \nabla h = 0
\]

\[
\partial_t h + \nabla \cdot (h \mathbf{u}) = 0
\]

- Rossby number \( \varepsilon = U/(fL) \ll 1 \)
- Burger number \( B_u = gH/(f^2 L^2) \)

Semi-geostrophic scaling (aka. Phillips type 2 scaling/frontal geostrophic regime):

\( B_u = \varepsilon \)

(Quasi-geostrophic regime is \( B_u = O(1) \) with \( h = 1 + O(\varepsilon) \); not considered here.)

Eliassen/Hoskins: geostrophic momentum approximation

\[
\varepsilon (\partial_t + \mathbf{u}_\varepsilon \cdot \nabla) \nabla^\perp h_\varepsilon + \mathbf{u}_\varepsilon^\perp + \nabla h_\varepsilon = 0
\]

- Canonical Hamiltonian system
- Adveceted PV in geostrophic coordinates (Hoskins, 1975)
2.2. Example ctd.: First order balance models

\[ L_\varepsilon = \int h_\varepsilon \left( \mathbf{R} \cdot \mathbf{u}_\varepsilon + \frac{1}{2} \varepsilon |\mathbf{u}_\varepsilon|^2 - \frac{1}{2} h_\varepsilon \right) \, d\mathbf{x} \]

where \( \nabla^\perp \cdot \mathbf{R} = f \equiv 1 \)

**Expansion in \( \varepsilon \)**

\[ L_\varepsilon = \int h \left( \mathbf{R} \cdot \mathbf{u} - \frac{1}{2} h \right) \, d\mathbf{x} + \varepsilon \int h \left( \mathbf{v}^\perp \cdot \mathbf{u} + \frac{1}{2} |\mathbf{u}|^2 + \frac{1}{2} h \nabla \cdot \mathbf{v} \right) \, d\mathbf{x} + O(\varepsilon^2) \]

**Degeneracy condition**

\[ \mathbf{v} = \frac{1}{2} \mathbf{u}^\perp + \lambda \nabla h \]

**Salmon's (1985) \( L_1 \)-model:**

- Any balance model will have \( \mathbf{u} = \nabla^\perp h + O(\varepsilon) \), so \( \lambda = \frac{1}{2} \) implies \( \mathbf{v} = O(\varepsilon) \)
- Forget the transformation!
2.3. First order model dynamics

Set $\sigma = \varepsilon(\lambda + \frac{1}{2})$. Then

$$\partial_t q + u \cdot \nabla q = 0$$

$$(q - \sigma \Delta) h = f$$

$$(1 - \sigma (h \Delta + 2 \nabla h \cdot \nabla)) u = \nabla \perp \left[ h - \varepsilon \lambda (2 h \Delta h + |\nabla h|^2) \right]$$

What is known:

- Solution theory: Çalık, O., Vasylkevych (2013)
- Numerically well-behaved models, consistent initialization is difficult: Dritschel, Gottwald, O. (WIP)
- Justification: open
2.4. The bigger picture

Semigeostrophic equations

- Solution theory: Cullen, Purser, Gangbo, Feldman, … (1980s–today)
- Justification: open

Generalizations

- Spatially varying Coriolis parameter: O., Vasylkevych (2013)
- Stratified models: O., Vasylkevych (2013)
- Quasigeostrophic scaling, higher order models: O. (2006)

Beyond fluids

- The semilinear Klein–Gordon equation (to follow)
- Analysis of variational time-integrators
3. PDE case study: semilinear Klein–Gordon equation

\[
\frac{\hbar^2}{2mc^2} \dddot{\Psi} - \frac{\hbar^2}{2m} \Delta \Psi + \frac{mc^2}{2} \dot{\Psi} + f(|\Psi|^2) \Psi = 0
\]

Modulated wave function

\[
\psi = \Psi e^{\frac{imc^2t}{\hbar}}
\]

Then

\[
i\hbar \dot{\psi} - \frac{\hbar^2}{2mc^2} \dddot{\psi} + \frac{\hbar^2}{2m} \Delta \psi - f(|\psi|^2) \psi = 0
\]

Non-relativistic limit \( c \to \infty \)

- Convergence to NLS in energy space (Machihara, Nakanishi, Ozawa, Masmudi, 2000s)

- Structurally a “semigeostrophic” limit

- Can we use variational methods to derive a hierarchy of “balance models” for slow motion in the weakly relativistic regime?
3.1. Setup

Lagrangian (non-dimensionalized)

\[ L(u, \dot{u}) = \int_T \left( \frac{\varepsilon}{2} |\dot{u}|^2 + \frac{i}{2} \dot{u} \bar{u} - \frac{1}{2} |u_x|^2 + V(u, \bar{u}) \right) dx \]

Full model as first order system

\[ \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta/\varepsilon & i/\varepsilon \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(u)/\varepsilon \end{pmatrix} \]

Notation: \( g(u) = f(|u|^2)u \) where \( V(u, \bar{u}) = \frac{1}{2} F(|u|^2) \) with \( F' = f \).

Eigenoperators of linear part

\[ L_\pm = i \frac{1 \pm \sqrt{1 - 4\varepsilon \Delta}}{2\varepsilon} \]

Anatz for fast variable – remove linear slow motion to all orders

\[ w = v - iL_- u - F_{\text{slow}}^{n+1}(u) \]
3.2. Recurrence relation for slow vector field

Slow-fast splitting

\[ \dot{u} = iL_- u + F_{\text{slow}}^n(u) + \varepsilon^{n+1} f_{n+1}(u) + w \]
\[ \dot{w} = \left( \frac{i}{\varepsilon} - iL_- - DF_{\text{slow}}^{n+1}(u) \right) w + \frac{1}{\varepsilon} \left( g(u) + i(1 - \varepsilon L_-) F_{\text{slow}}^{n+1}(u) \right) \]
\[ - iDF_{\text{slow}}^{n+1}(u) L_- u - DF_{\text{slow}}^{n+1}(u) F_{\text{slow}}^{n+1}(u) \]

Construction of the slow vector field

\[ M^{-1} \equiv 1 - \varepsilon L_- = \frac{1 + \sqrt{1 - 4\varepsilon\Delta}}{2} \]

is positive, self-adjoint, first-order with compact inverse \( M \). Thus,

\[ f_0(u) = iMg(u) \]
\[ f_{\ell+1} = M \left( Df_{\ell}(u)L_- u - i \sum_{j+k=\ell} Df_j(u)f_k(u) \right) \]

No recurrent loss of regularity! Persistence of \( L^2 \)-smallness of \( w \) can be achieved to any order in \( \varepsilon \) uniformly in the regularity class of the initial data.
3.3. Variational asymptotics for linear Klein–Gordon

Quadratic action functional

\[ S_\varepsilon = \frac{\varepsilon}{2} \langle T^2 u_\varepsilon, u_\varepsilon \rangle + \frac{1}{2} \langle Tu_\varepsilon, u_\varepsilon \rangle + \frac{1}{2} \langle \Delta u_\varepsilon, u_\varepsilon \rangle. \]

where \( \langle \cdot, \cdot \rangle \) is the space-time inner product and \( T \equiv i \frac{\partial}{\partial t} \) is formally self-adjoint.

Degeneracy condition: Can we choose \( u_\varepsilon = \phi(\varepsilon T, \varepsilon \Delta) u \) such that

\[ S_\varepsilon = \frac{1}{2} \langle (\varepsilon T^2 + T + \Delta) \phi^2(\varepsilon T, \varepsilon \Delta) u, u \rangle = \frac{1}{2} \langle (T + \Delta \theta(\varepsilon \Delta)) u, u \rangle \]

I.e., find generating functions \( \phi(\xi, \eta) \) and \( \theta(\eta) \), analytic near the origin, with

\[ (\xi^2 + \xi + \eta) \phi^2(\xi, \eta) = \xi + \eta \theta(\eta) \]

It can be shown that there is a unique choice, namely

\[ \theta(\eta) = \frac{1 - \sqrt{1 - 4\eta}}{2\eta} \quad \text{and} \quad \phi(\xi, \eta) = \frac{\sqrt{k(\eta)}}{\sqrt{1 + \xi k(\eta)}} \quad \text{with} \quad k(\eta) = \frac{2}{1 + \sqrt{1 - 4\eta}} \]
3.4. Expansion of the linear transformation

- When plugging the linear transformation into the potential, we need to expand
- Can we do this without losing derivatives?

**Naive expansion**

Let $K$ be the compact operator with symbol $k$. Then

$$
\phi(\varepsilon T, \varepsilon \Delta) = \frac{\sqrt{K}}{\sqrt{1 + \varepsilon TK}} = \sqrt{K} \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^j \varepsilon^j (TK)^j
$$

On solutions of the slow equation, $TK$ is a zero order operator, but its operator norm is $O(\varepsilon^{-1})$ unless we lose derivatives.

**Better expansion – use lower order “balance”**

$$
\phi(\varepsilon T, \varepsilon \Delta) = \frac{\sqrt{K}}{\sqrt{1 + \varepsilon TK}} = \frac{\sqrt{M}}{\sqrt{1 + \varepsilon (T + L_-)M}} \quad \text{with} \quad M = \frac{1}{\sqrt{1 - 4\varepsilon \Delta}}
$$

On solutions of the slow equation, $(T + L_-)M$ is of zero order uniformly!
3.5. Shadowing theorem

Let \( u \) denote a solution of the slow Euler–Lagrange equation \( u(0) \in H^2 \). Let \( u_\varepsilon \) solve the full Euler–Lagrange equation consistently initialized via

\[
\begin{align*}
  u_\varepsilon(0) &= \Phi_n[u]\bigg|_{t=0} \\
  \dot{u}_\varepsilon(0) &= \frac{d}{dt} \Phi_n[u]\bigg|_{t=0}
\end{align*}
\]

Then for every fixed \( T > 0 \) there exist \( \varepsilon_0 > 0 \) and \( c = c(u(0), T) \) such that for all \( 0 < \varepsilon \leq \varepsilon_0 \),

\[
\sup_{t \in [0,T]} \| u_\varepsilon(t) - \Phi_n[u(t)] \|_{L^2} \leq c \varepsilon^{n+1}
\]

**Proof**

Note that all operators are bounded – proceed as in finite dimensions.

**Conclusion**

*Hamiltonian PDE asymptotics without “loss of derivatives”*