Cascades of Scales in the Atmosphere: Theory, Numerics, and Data-Based Modelling

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Issues with “Cascades of Scales” — CRC 1114

Scale-dependent balances and data assimilation

Data-based modelling for turbulent flows

Energy transfer in a hurricane

Summary
Cascades of Scales ⇔ "more than two scales"
A “root model”, i.e.,

A comprehensive (mathematical) description of all participating processes on all scales
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A comprehensive (mathematical) description of all participating processes on all scales may or may not be available as a mathematical model, computer code, natural language description.
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**computer code**, 
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A “root model”, i.e.,

A comprehensive (mathematical) description of all participating processes on all scales may or may not be available as a mathematical model, computer code, natural language description.
Data:

- observational or experimental
- “simulational”

needed to “fill in the gaps”.
CRC 1114 Research Areas

- Mode index $k$
- Length, time, mode index $n$
- Amplitude or frequency

Diagram:
- Large scales
- Small scales
- No of degrees of freedom
- Mode index $k$
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**Scale-dependent balances and data assimilation**

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Summary
Multiscale data and asymptotic model assimilation for atmospheric flows

• Bayesian approaches to observation & model filtering

• Scale-dependent balances of the NWP + DA composite model

• Numerics for balanced DA-corrections
Initial data: “assimilation” of a hot bubble

\[ \delta \theta = 2 \text{ K}; \quad \left[-10, 10\right] \times \left[0, 10\right] \text{ km} \]

\[ \delta \pi \text{ at } x 10^{-4} \]

compressible \( \delta \pi \sim 10^{-4} \)

\[ \alpha \equiv 1.0 \]
Pseudo-incompressible / compressible numerics

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) + P \nabla \pi = -\rho g \mathbf{k}
\]

\[
\alpha P_t + \nabla \cdot (P \mathbf{v}) = 0
\]

\[
\pi = P^{\gamma^{-1}}
\]

\[
(P = \rho \theta)
\]

Seamless blending of solvers for \( \alpha \in [0, 1] \) enables

Model comparison based on “identical” numerics

Generation of balanced initial data
Balanced “assimilation” of a hot bubble

\[ \delta \theta = 2 \text{ K}; [-10, 10] \times [0, 10] \text{ km} \]

\[ \theta \]

\[ \delta \pi \text{ at } \bullet \]

\[ \alpha \equiv 1.0 \]

compressible \( \delta \pi \sim 10^{-4} \)

blended \( \delta \pi \sim 10^{-7} \)

\[ \alpha = [0.0, 0.733, 1.0, 1.0, \ldots] \]
Predictor

Solve auxiliary hyperbolic system over \( t^n \rightarrow t^{n+1} \)
(by your favorit 2nd order scheme)*

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \circ \mathbf{v}) &= -\rho g \mathbf{k} - P \nabla \pi^n \\
P_t + \nabla \cdot (P \mathbf{v}) &= 0
\end{align*}
\]

Predicted scalars already have desired accuracy

\[
\begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{n+1,*} = \begin{pmatrix} \rho \\ P \\ \theta \end{pmatrix}^{n+1} + O((\Delta t)^3)
\]
Corrector for $P$-fluxes (for stability)

\[ \pi^{n+1} = \pi^n + \delta \pi \]

\[
(P\nu)^{n+1/2} = (P\nu)^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta \pi
\]

\[
P^{n+1} = P^{n+1,*} + \frac{(\Delta t)^2}{2} \nabla \cdot (P\theta \nabla \delta \pi)
\]

with

\[
P^{n+1} = P^n \equiv \overline{P}(z) \quad \text{pseudo-incompressible}
\]

\[
P^{n+1} = P^n + \left( \frac{\partial P}{\partial \pi} \right)^* \delta \pi \quad \text{compressible}
\]

\[
P^{n+1} = P^n + \alpha \left( \frac{\partial P}{\partial \pi} \right)^* \delta \pi \quad \text{blended model}
\]
Corrector for $P$-fluxes (for stability)

$$\pi^{n+1} = \pi^n + \delta\pi$$

$$\left(P\nu\right)^{n+1/2} = \left(P\nu\right)^{n+1/2,*} - \frac{\Delta t}{2} P\theta \nabla \delta\pi$$

$$P^{n+1} = P^{n+1,*} + \frac{\left(\Delta t\right)^2}{2} \nabla \cdot \left(P\theta \nabla \delta\pi\right)$$

Elliptic pressure equation

$$\frac{2\alpha}{\left(\Delta t\right)^2} \left(\frac{\partial P}{\partial \pi}\right)^{n+1/2} \delta\pi - \nabla \cdot \left(P\theta \nabla \delta\pi\right) = \frac{2}{\left(\Delta t\right)^2} \left(P^{n+1,*} - P^n\right)$$

* K., TCFD, 23, 161–195, (2009); Benacchio et al., MWR, online (2014)
Balanced “assimilation” of a hot bubble

\( \delta \theta = 2 \text{ K}; \ [-10, 10] \times [0, 10] \text{ km} \)

\( \theta \)

\( \delta \pi \text{ at } \bullet \)

compressible \( \delta \pi \sim 10^{-4} \)

\( \alpha \equiv 1.0 \)

blended \( \delta \pi \sim 10^{-7} \)

\( \alpha = [0.0, 0.733, 1.0, 1.0, ...] \)

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**Data-based modelling for turbulent flows**

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Summary
Multiscale tensor decomposition methods for partial differential equations

- **Data-based reduced modelling of turbulence data**
- Self-similarity via tensor product decompositions
- LES closure
  - reduced model from resolved scales only
  - extrapolation in scale & averaging ⇒ LES closure

*Yserentant & Schneider: general PDE-solvers based on tensor product schemes*
Stencil-conditioned subgrid scale modelling of flux corrections $f$

\[
f_{t,x} = \mu(t, x) + A(t, x) \phi_1(f_{t-\tau}, \ldots, f_{t-m\tau})x + B(t, x) \phi_2(u_{t,x}) + \epsilon_{t,x}.
\]

**FEM-BV-VARX**

**ILES vs. ExpLES vs. linear dissipation:**  ($\phi_1$: linear; $u$: coarse-grid stencil data)

\[
B \phi_2(u) = (b_1 f^{1st} + b_2 f^{2nd} + b_3 f^{WENO} + b_S f^{Smagorinsky} + b_{lin} f^{lin}) (u)
\]
\[
f_{t,x} = \mu(t, x) + A(t, x)\phi_1(f_{t-\tau}, \ldots, f_{t-m\tau}, x) + B(t, x)\phi_2(u_{t,x}) + \epsilon_{t,x}.
\]

- represent \((\mu, A, B) (t, x)\) by \(K\) sets of \(k\) parameters \(\{\Theta_j \equiv (\Theta_1, \ldots, \Theta_k)\}_{j=1}^K\) \((K \cdot k)\)

- using model affiliations \(\gamma_j(t, x) \in [0, 1]\) with \(\sum_{j=1}^K \gamma_j(t, x) = 1\)

- represent \(\gamma_j(t, x)\) by Finite Element Methods in time with Bounded Variation

\[
TV_t (\gamma_j(t, x)) \leq C \quad (N_x \cdot (K - 1) \cdot C)
\]

- given \((k, K, C)\), minimize the interpolated distance

\[
\int_t \int_x \delta_{t,x} \, dx \, dt \rightarrow \min_{\gamma, \Theta} \quad \text{where} \quad \delta_{t,x} = \sum_{j=1}^K \gamma_j (t, x) \| \epsilon_{t,x}^{\Theta_j} \|
\]

- pick the model that minimizes a modified Akaike’s information criterion

\[
\text{mAIC} = 2(n + K \cdot k + C \cdot (K - 1) \cdot N_x) - 2 \ln \left( \text{Likelihood} \left( \delta_{t,x} \mid (\gamma, \Theta)_{k,K,C} \right) \right)
\]

based on posterior \(n\)-moment model for probability distributions of the scalar \(\delta_{t,x}\) \((n)\)

for a review see Metzner, Putzig, Horenko (2012), CAMCoS, 7, 179–229
Fully developed channel flow: testing ILES
data coarsened to $50 \times 50 \times 50$ cells

Three regimes:
“core flow”, “transition”, “wall model”
Fully developed channel flow: testing ILES
data coarsened to $50 \times 50 \times 50$ cells

Flux corrections from FEM-BV-VARX-Model

Fully developed channel flow: testing ILES data coarsened to $25 \times 25 \times 25$ cells

Individual candidate flux corrections

Fully developed channel flow: testing ILES data coarsened to $25 \times 25 \times 25$ cells

Flux corrections from FEM-BV-VARX-Model
Recent developments

- Extension of FEM-BV-Information theory framework to testing causality (I. Horenko, S. Gerber, (2014,2015))

- Applications to stable boundary layer observations (N. Vercauteren, R.K., JAS (2014))
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Summary
Multiscale structure of atmospheric vortices

- Asymptotics of multiple radial/vertical layers
- Small-scale convection
  - asymptotic structure
  - determining factors
  - stochastic appearance
Tropical easterly african waves
Developing tropical storm
(streamlines in co-moving frame and Okubo-Weiss-parameter (color))

\[ \mathcal{R}_o = \frac{|v|}{fL} \sim \frac{1}{10} \]
Developed hurricane

\[ R_{mw}^* \approx 50 \ldots 200 \text{ km} \]

\[ u_\theta \approx 30 \ldots 60 \text{ m/s} \]

\( R_{mw} \): radius of max. wind

Hurricane "Rita"

\[ R_0 = \frac{u_{\theta,\text{max}}}{f R_{mw}} \sim 10 \]
Radial momentum balance regimes

\[- \frac{1}{\rho} \frac{\partial p}{\partial r} + f u_\theta = \mathcal{O}(1) \quad \text{geostrophic} \quad \text{Ro} \ll 1 \quad \text{typical “weather”} \]

\[\frac{u_\theta^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial r} + f u_\theta = \mathcal{O}(1) \quad \text{gradient wind} \quad \text{Ro} = \mathcal{O}(1) \quad \text{tropical storm incipient hurricane} \]

\[\frac{u_\theta^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial r} = \mathcal{O}(1) \quad \text{cyclostrophic} \quad \text{Ro} \gg 1 \quad \text{hurricane} \]
Vortex tilt in the incipient hurricane stage
(Velocity potential)

200 hPa
(∼ 12 km)

925 hPa
(∼ 0.8 km)

∼ 200 km
Asymptotic scaling regime

\[ t_{\text{syn}} = \frac{h_{\text{sc}}}{u_{\text{ref}}} \delta^2; \quad L_{\text{syn}} = \frac{h_{\text{sc}}}{\delta^2}; \quad |\mathbf{v}_\parallel| = \mathcal{O}(1); \]

farfield: classical QG theory

\[ |\mathbf{v}_\parallel| L = \mathcal{O}(\delta^{-2}); \quad |\mathbf{v}_\parallel|/fL = \mathcal{O}(\delta) \]

core: gradient wind scaling

\[ L_{\text{mes}} = \frac{h_{\text{sc}}}{\delta^{3/2}}; \quad |\mathbf{v}_\parallel| = \mathcal{O}\left(\frac{1}{\delta^{1/2}}\right) \]

\[ |\mathbf{v}_\parallel| L = \mathcal{O}(\delta^{-2}); \quad |\mathbf{v}_\parallel|/fL = \mathcal{O}(1) \]
Result of matched asymptotic expansion analysis:

3D Theory for

vortex motion, vortex core dynamics*,
and the role of subscale moist processes*

* Includes strong vortex tilt

* Modelled by prescribed heating patterns here
Dominant balances
(0th & 1st circumferential Fourier modes: \( w = w_0 + w_{11} \cos \theta + w_{12} \sin \theta + \ldots \))

gradient wind balance (0th) and hydrostatics (1st) in the tilted vortex

\[
\frac{1}{\rho} \frac{\partial p}{\partial r} = u_\theta \left( \frac{u_\theta}{r} + f \right), \quad \Theta_{1k} = -\frac{1}{\rho} \frac{\partial}{\partial r} \frac{\partial}{\partial z} \left( e_r \cdot \hat{X} \right)_{1k} \]

potential temperature transport (1st)

\[
-\frac{u_\theta}{r} \Theta_{1k^*} + w_{1k} \frac{d\Theta}{dz} = Q_{\Theta,1k} \quad (k^* = 3 - k)
\]

1st-mode phase relation: vertical velocity – diabatic sources & vortex tilt

\[
w_{1k} = \frac{1}{\frac{d\Theta}{dz}} \left[ Q_{\Theta,1k} + \left( e_r \cdot \frac{\partial \hat{X}}{\partial z} \right)_{k} \left( \frac{u_\theta}{r} \left( \frac{u_\theta}{r} + f \right) \right) \right]
\]
Evolution of primary circulation

\[
\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00} \left( \frac{\partial u_{\theta}}{\partial r} + \frac{u_{\theta}}{r} + f \right) = -u_{r,*} \left( \frac{u_{\theta}}{r} + f \right)
\]

standard axisymmetric balance

\[u_{r,*} = \left\langle w \frac{\partial}{\partial z} \left( e_r \cdot \mathbf{\hat{X}} \right) \right\rangle_{\theta}\]

\[e_r \cdot \mathbf{\hat{X}} = \mathbf{\hat{X}} \cos \theta + \mathbf{\hat{Y}} \sin \theta\]

\[w_{1k} = \frac{1}{d\Theta/dz} \left[ Q_{\Theta,1k} + \frac{\partial}{\partial z} \left( e_r \cdot \mathbf{\hat{X}} \right) \right]_k \frac{u_{\theta}}{r} \left( \frac{u_{\theta}^2}{r} + f u_{\theta} \right)\]
Asynchronous heating ⇒ kinetic energy of primary circulation

\[
\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00} \left( \frac{\partial u_{\theta}}{\partial r} + \frac{u_{\theta}}{r} + f \right) = -u_{r,*} \left( \frac{u_{\theta}}{r} + f \right)
\]

standard axisymmetric balance

\[
u_{r,*} = \left\langle w \frac{\partial}{\partial z} \left( e_r \cdot \hat{\mathbf{X}} \right) \right\rangle_\theta = \frac{1}{d\Theta/dz} \left( Q_{\Theta,11} \frac{\partial \hat{X}}{\partial z} + Q_{\Theta,12} \frac{\partial \hat{Y}}{\partial z} \right)
\]

\[
e_r \cdot \hat{\mathbf{X}} = \hat{X} \cos \theta + \hat{Y} \sin \theta
\]

\[
w_{1k} = \frac{1}{d\Theta/dz} \left[ Q_{\Theta,1k} + \frac{\partial}{\partial z} \left( e_r \cdot \hat{\mathbf{X}} \right)_k \frac{u_{\theta}}{r} \left( \frac{u_{\theta}^2}{r} + f u_{\theta} \right) \right]
\]
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