The influence of fast waves and fluctuations on the evolution of slow solutions of the Boussinesq Equations

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“Low Rossby limiting dynamics for stably stratified flows”, B. Wingate, P. Embid, M. Holmes-Cerfon, M. Taylor, *Journal of Fluid Mechanics*, 2011

“An Asymptotic Parallel-in-Time Method for Highly Oscillatory PDEs”, T. Haut, B. Wingate, **38**, 2, *SIAM Journal of Scientific Computing*, 2014

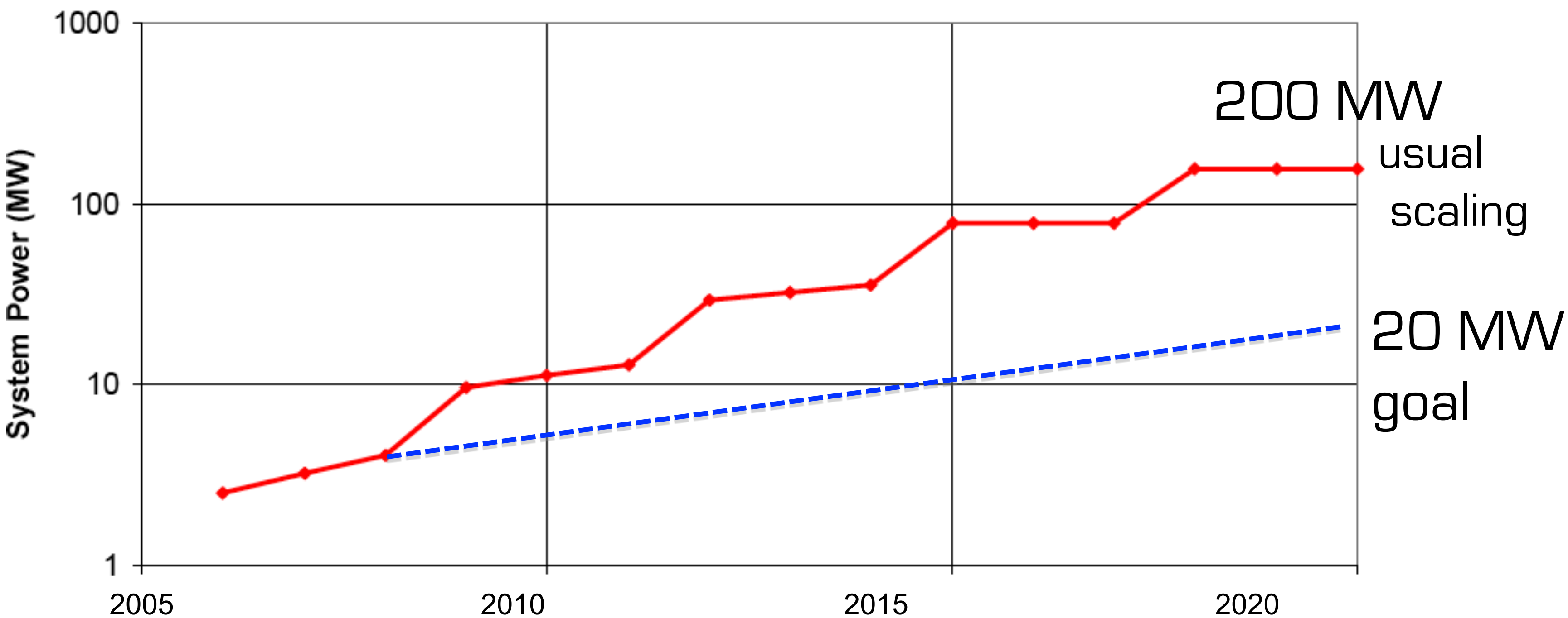
“The influence of fast waves and fluctuations on the evolution of the slow manifold”, J. Whitehead and B. Wingate, *submitted to Journal of Fluid Mechanics*, 2014

“The effect of two distinct fast time scales in the rotating, stratified Boussinesq equations”, J. Whitehead, T. Haut, B. Wingate, *submitted to Nonlinearity*, 2014

Outline

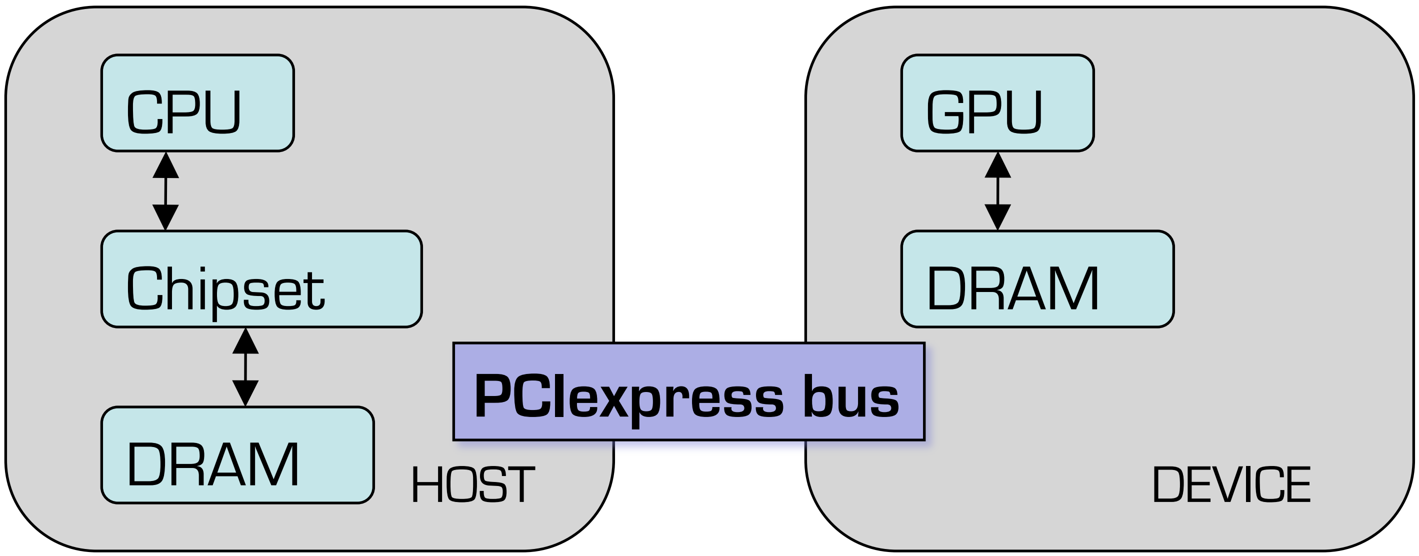
- High performance computing in the next decade
- Implications for physics with oscillatory stiffness
  (including numerical models of weather and climate)
- The time barrier
  (back to Leith, Lorenz, Charney?)
- The influence of fast waves and fluctuations of the evolution of the slow dynamics in the Boussinesq system – three distinguished limits
- Summary
- Some interesting problems

For new computing architectures power is limiting factor



A US city of 80,000 people is estimated to be 45 MW. The state of New York 140 GW. Exascale design for 1 exaflop to be 20 MW instead of 200MW.

Example of the CPU + GPU configuration



Data parallel pieces of an algorithm are executed on the device as KERNELS that are executed as many THREADS grouped together in BLOCKS. BLOCKS grouped together in GRIDS. Each concept has hardware meaning.

Data parallel

- The key to getting performance through the use of co-processors (such as the GPU) is the idea of Data-Parallel Computing
- Identical operations executed on many data elements in parallel (simplified flow control allows increased ratio of compute logic to control logic)

DATA INDEPENDENT	DATA DEPENDENT
a = 2 * pi + x b = y * z	a = 2 * pi + x b = y * a

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Implications of changing computing architectures on physics of climate and weather models

- The speed of computer processors is not going to increase as fast as they used to. But there will be a lot more of them and they will have unprecedented (billion-way) parallelism.
- Because current algorithms need to reduce their maximum time step as the number of grid points increases (due to the oscillatory stiffness), these new machines may not reduce wall clock time . You may be able to have a higher resolution grid but you will wait longer for a simulation to complete.
- From a wall-clock-time point of view, current models may appear to dissipate into the machine – this is ***the time barrier***
- Fault tolerance/resistance (large numbers of processors)
- Asynchronous algorithms – wasting time waiting for one time step to finish?

Algorithms and new computing architectures

- **Fixed grid on N processors**

  For a fixed grid you may have an optimal distribution of the grid on N processors, then the only dimension left available for parallelization is time.

- **Grid refinement** (we'll still have to wait for each time step)

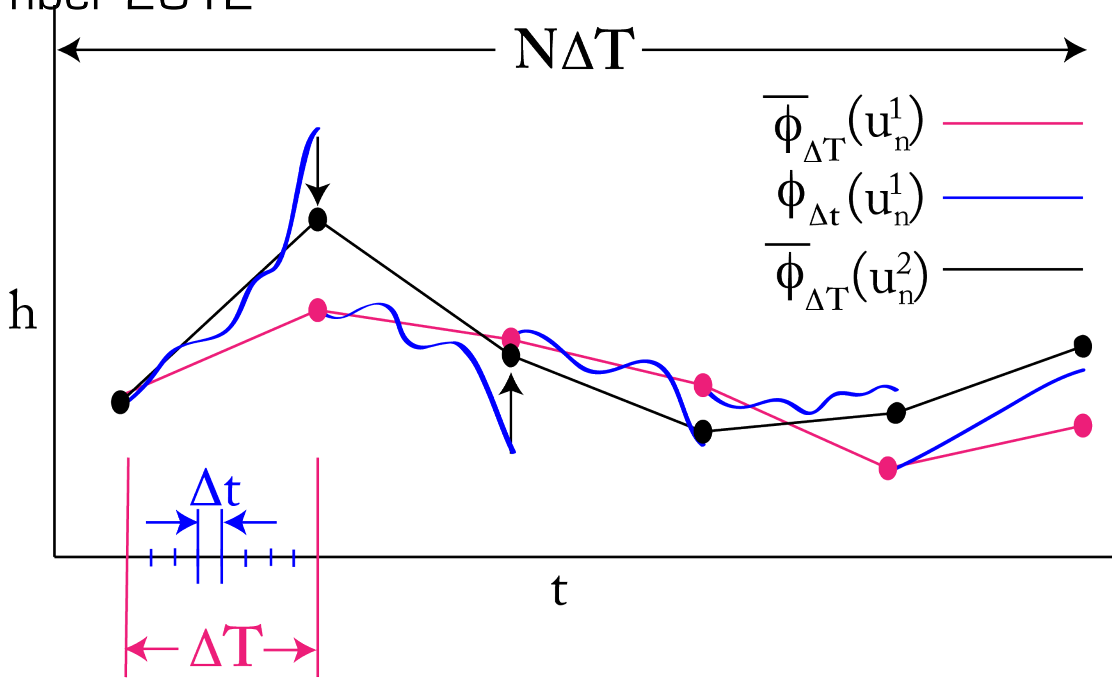
  Because current algorithms need to reduce their maximum time step as the number of grid points increases, these new machines may not significantly reduce wall clock time . You may be able to have a higher resolution grid but you will still wait a longer time for each time step to complete.

  From a wall-clock-time point of view, current models may appear to dissipate into the machine.

- **Fault tolerance/resistance** (large numbers of processors)
- **Asynchronous algorithms** – wasting time waiting for one time step to finish?

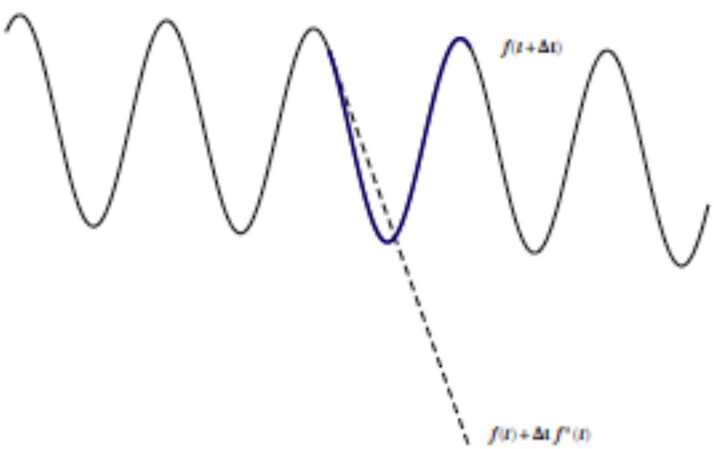
The time barrier – parallel-in-time (or Parareal)

- The numerics community is pretty good at doing parallelism in space (domain decomposition) and will get even better
- Nievergelt (1964)
- Lions, Maday, Turinici, (2001) 'Parareal', it has been successful for problems with dissipative stiffness
- A good introduction to these ideas can be found on the Newton Institute web site by Jean Côté under the AMM program in September 2012

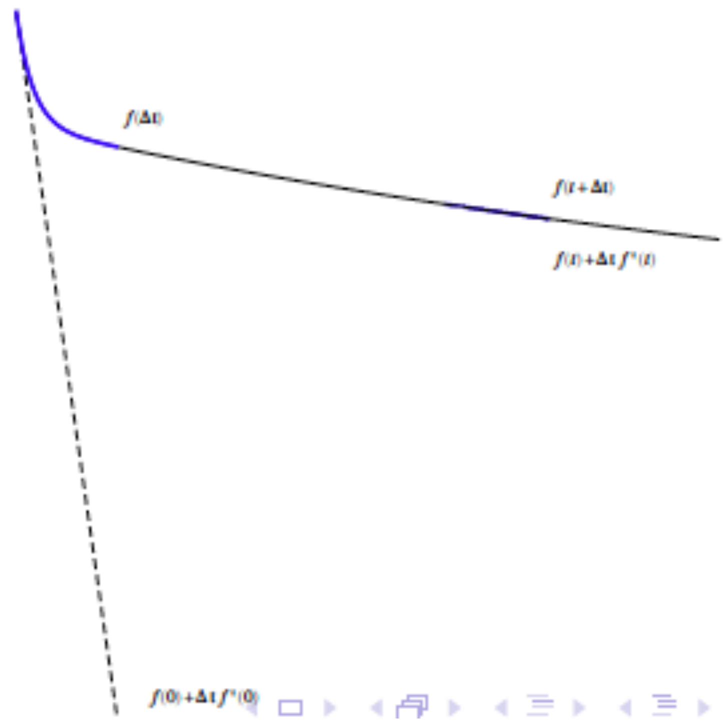


Stiffness in Parareal – many successes with dissipative stiffness but not oscillatory stiffness

(a) Stiffness from oscillations



(b) Stiffness from dissipation



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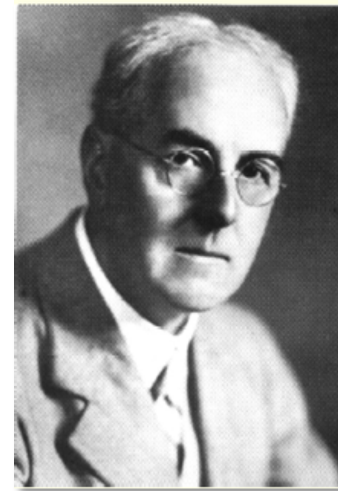
Slow Dynamics

- L.F. Richardson in (1922) - using 'computers' (**us**)
- Charney (1948) and Charney (1950) - derived 'slow' or Quasi-Geostrophic (QG) equations
- Charney and Phillips (1953) – the first realistic numerical weather prediction using the QG eqs



L.F. Richardson

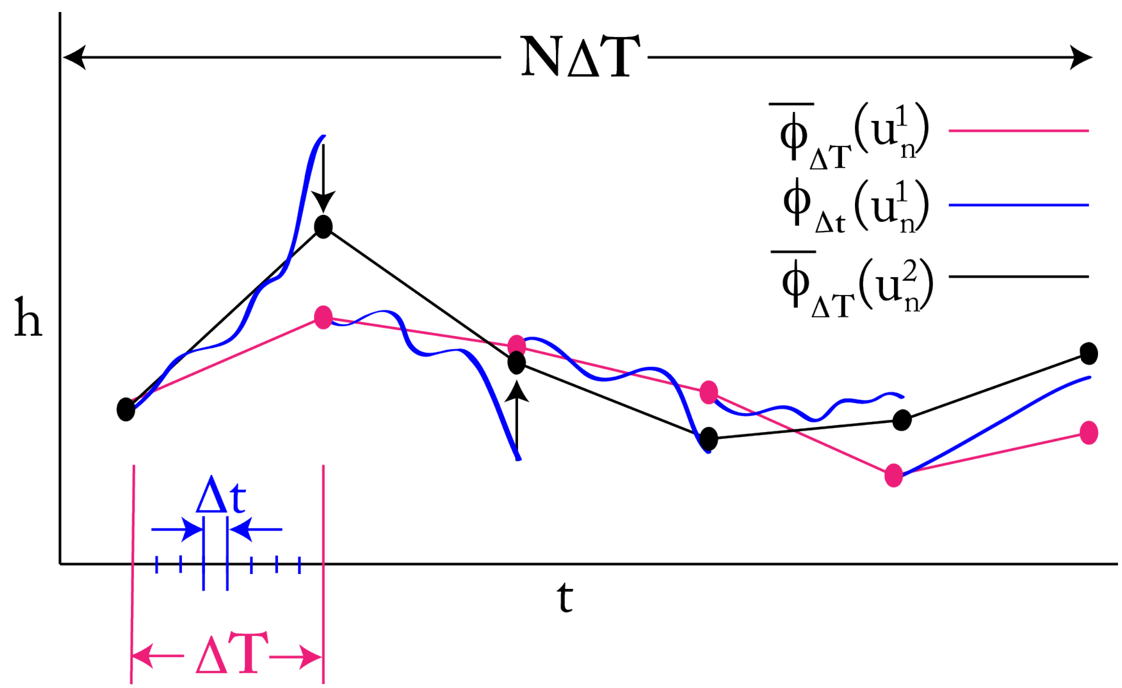
Chaney, Phillips, Lewis, Gilbarg & Platzman in front of the 'MANIAC I'

J. von Neumann

J. Charney

An *asymptotic* parareal method was successful in 2013

- With oscillatory stiffness making a good approximation for large time steps is difficult.
- Using a reduced equation for a 'slow' guess was mentioned early in the parareal literature but hasn't been very successful for oscillatory stiffness.



"An Asymptotic Parallel-in-Time Method for Highly Oscillatory PDEs", T. Haut, B. Wingate, *to appear in SIAM Journal of Scientific Computing*, 2014

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Why does the method work? Continued

Slow Manifolds (center manifolds, dynamical systems, etc)

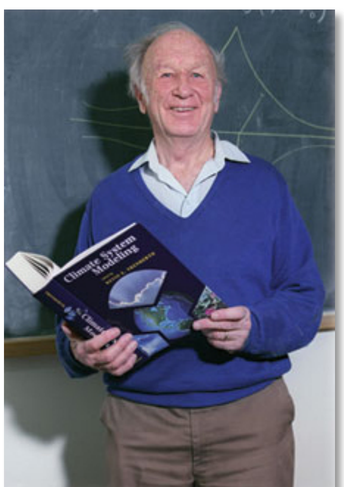
Leith, **Nonlinear Normal Mode Initialization and Quasi -Geostrophic Theory** (1980)

Lorenz, **On the Existence of a Slow Manifold** (1986)

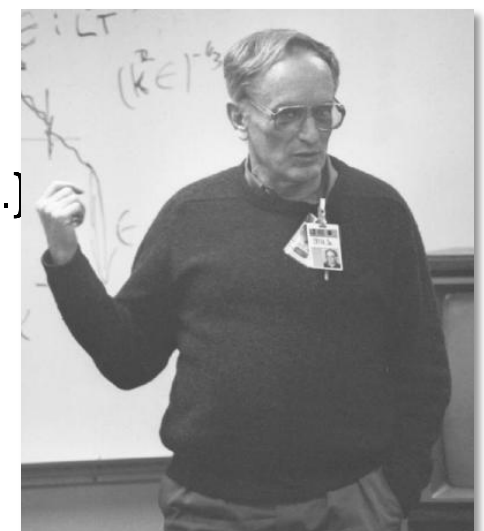
Lorenz and Krishnamurthy, **On the non-Existence of the Slow Manifold** (1987)

Lorenz, **The Slow Manifold – what is it**? (1991)

(....so far the answer is ‘it is a fuzzy manifold’.)

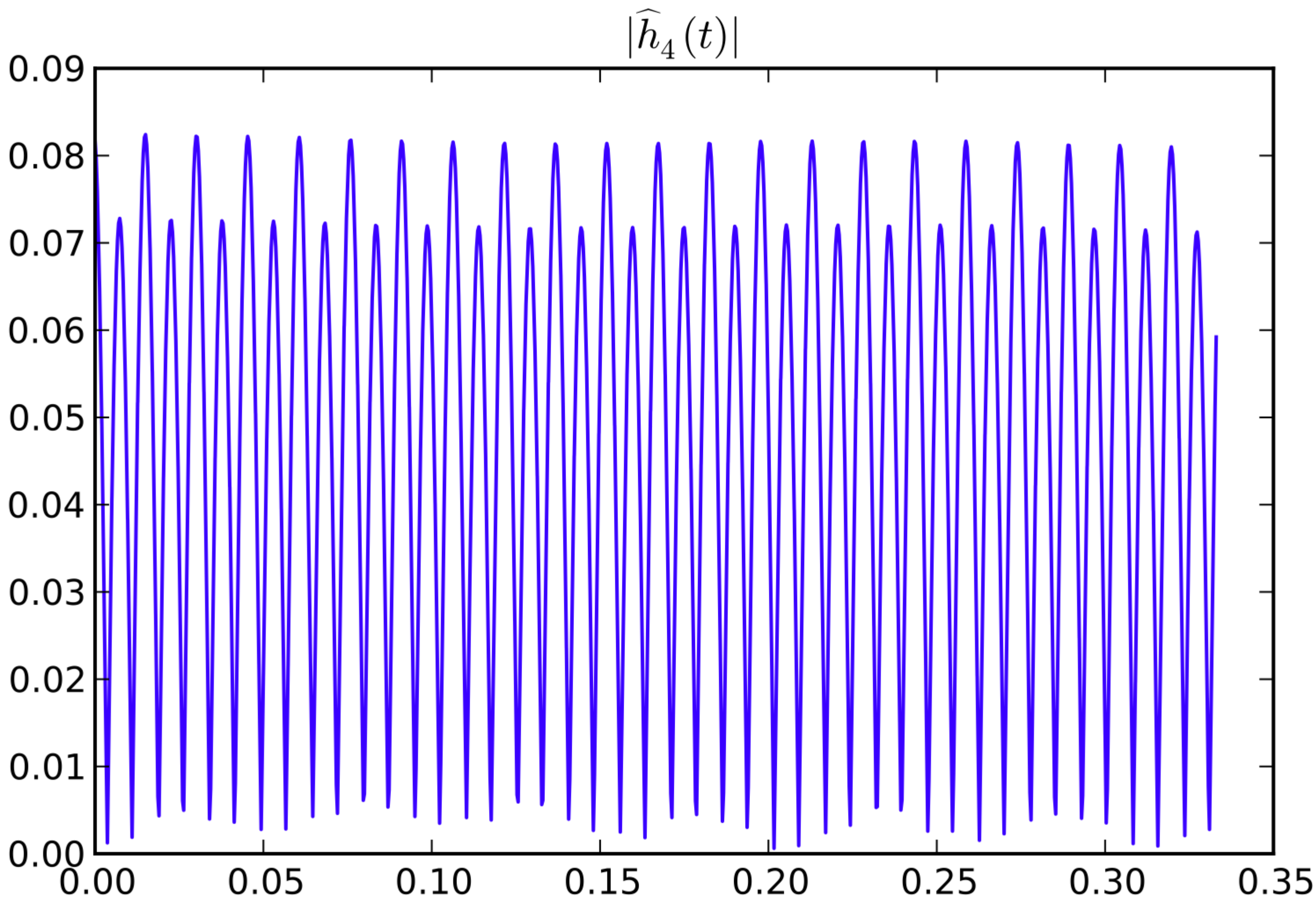


Ed Lorenz

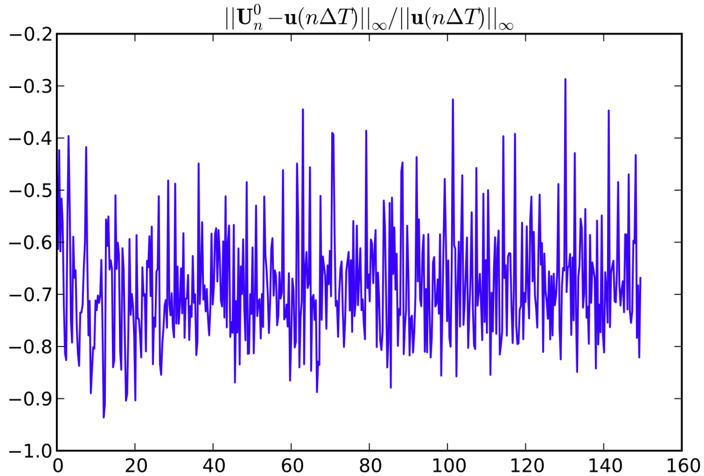


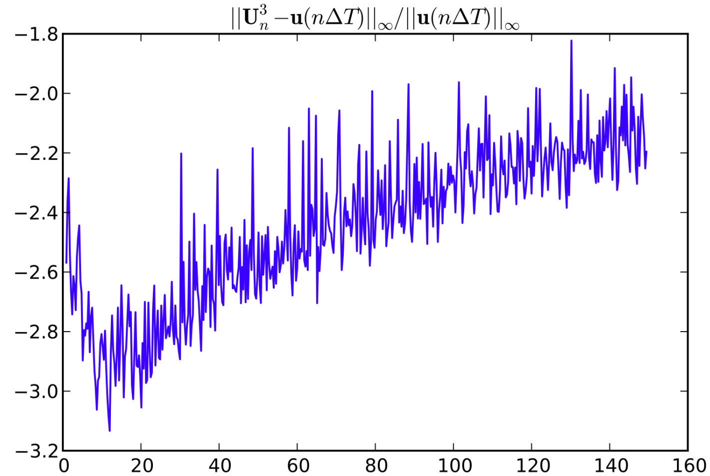
Chuck Leith

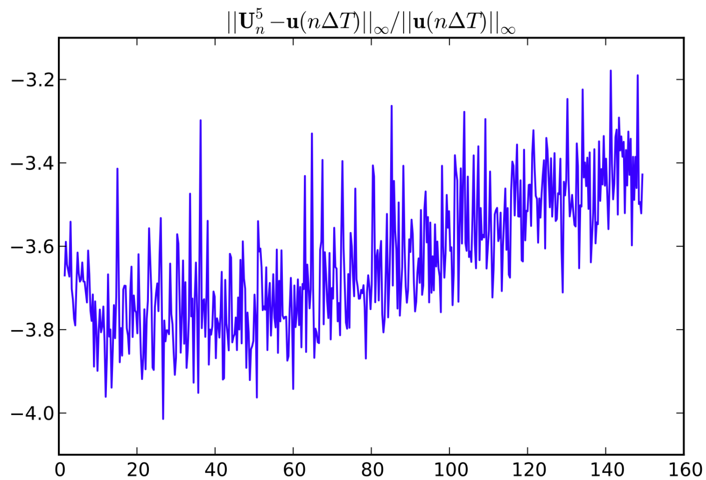
Example of the solution in time: epsilon=.01, Delta T = .3



Relative L^{∞} errors for epsilon=10^{-1} , using the asymptotic based parallel -in-time integrator and taking Delta T=3/10 and Delta t=1/500 . Each graph is for fixed a iteration k (1,3,5), and the errors are plotted as a function of the time n Delta T , n=1..N . The errors are shown on a log10 scale.







Nonlocal form in a Hilbert Space

Embid and Majda, 1996, 1997

Schochet, 1994

Klainerman and Majda 1981

$$\mathbf{u} = \begin{pmatrix} \mathbf{v} \\ \rho \end{pmatrix}$$

Hilbert Space X of vector fields **u** in L^2 that are divergence free and equipped with the L^2 norm.

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{Ro} L_{Ro}(\mathbf{u}) + \frac{1}{Fr} L_{Fr}(\mathbf{u}) + B(\mathbf{u}, \mathbf{u}) = \frac{1}{Re} D(\mathbf{u})$$

$$\mathbf{u}|_{t=0} = \mathbf{u_0}(\mathbf{x})$$

$$L_{Ro}(\mathbf{u}) = \begin{pmatrix} \widehat{\mathbf{z}} \times \mathbf{v} + \nabla \Delta^{-1} \omega_3 \\ 0 \end{pmatrix} \quad L_{Fr}(\mathbf{u}) = \begin{pmatrix} \widehat{\mathbf{z}}\, \rho + \nabla \Delta^{-1} (\frac{\partial \rho}{\partial z}) \\ -w \end{pmatrix}$$

$$B(\mathbf{u}, \mathbf{u}) = \begin{pmatrix} \mathbf{v} \cdot \nabla \mathbf{v} - \nabla \Delta^{-1} (\nabla \cdot \mathbf{v} \cdot \nabla \mathbf{v}) \\ \mathbf{v} \cdot \nabla \rho \end{pmatrix} \quad D(\mathbf{u}) = \begin{pmatrix} \Delta \mathbf{v} \\ 1/Pr \Delta \rho \end{pmatrix}$$

Oscillatory Stiffness in the PDE

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\epsilon}\mathcal{L}\mathbf{u} + \mathcal{N}\left(\mathbf{u},\mathbf{u}\right) = \mathcal{D}\mathbf{u}, \quad \mathbf{u}\left(0\right) = \mathbf{u}_0,$$

- The $\epsilon^{-1}\mathcal{L}$ operator results in temporal oscillations on a time scale of $\mathcal{O}\left(\epsilon\right)$
- Standard numerical time-stepping methods must use time steps $\Delta t = \mathcal{O}\left(\epsilon\right)$
- Interesting things happen to with nonlinearity and slow/fast time scales

One way to view the problem (also used for exponential integrators)

$$\mathbf{u}(t) = e^{-t/\epsilon\mathcal{L}}\mathbf{v}(t)$$

$$\frac{\partial \mathbf{v}}{\partial t} + e^{t/\epsilon\mathcal{L}}\mathcal{N}\left(e^{-t/\epsilon\mathcal{L}}\mathbf{v}(t), e^{-t/\epsilon\mathcal{L}}\mathbf{v}(t)\right) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} = \mathcal{O}(1)$$

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = \mathcal{O}\left(\frac{1}{\epsilon}\right)$$

v varies slower than **u**, but **v** still has some oscillations

Method of Multiple Scales

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t}_{\text{slow}} + \frac{1}{\epsilon}\frac{\partial}{\partial t}_{\text{fast}}$$
$$= \frac{\partial}{\partial t} + \frac{1}{\epsilon}\frac{\partial}{\partial \tau}$$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}^0(\mathbf{x}, t, \tau) + \epsilon \boxed{\mathbf{u}^1(\mathbf{x}, t, \tau)}$$

To avoid secularity the second order term must be smaller than the leading order term.

$$|\mathbf{u}^1(\mathbf{x}, t, \tau)| = o(\tau)$$

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An asymptotic method-of-multiple scales in time (another way to derive Quasi Geostrophy is a singular perturbation in time):

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{\epsilon}\mathcal{L}\mathbf{u} = \mathcal{N}(\mathbf{u}) + \mathcal{D}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0,$$

Asymptotic solution looks like:

$$\mathbf{u}(t) = \exp(-t/\epsilon\mathcal{L})\,\overline{\mathbf{u}}(t) + \mathcal{O}(\epsilon)$$

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} = \overline{\mathcal{N}}(\overline{\mathbf{u}}) + \overline{\mathcal{D}}\overline{\mathbf{u}}, \quad \overline{\mathbf{u}}(0) = \mathbf{u}_0.$$

$$\overline{\mathcal{D}}\overline{\mathbf{u}}(t) = \lim_{T\to\infty}\frac{1}{T}\int_0^T \left(e^{s\mathcal{L}}\mathcal{D}e^{-s\mathcal{L}}\right)\overline{\mathbf{u}}(t)\,ds$$

$$\overline{\mathcal{N}}(\overline{\mathbf{u}}(t)) = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{s\mathcal{L}}\mathcal{N}\left(e^{-s\mathcal{L}}\overline{\mathbf{u}}(t)\right)ds.$$

Embid and Majda, 1996, 1998, Majda and Embid, 1998, Schochet, 1994, Klainerman and Majda 1981, Wingate, Embid,, Cerfon-Holme Taylor, 2011

The role of near-resonances

- In the Asymptotic Parallel-in-time method we are nearly resolving the near-resonances. This is the reason the method is converging for parallel-in-time

- Can we get any insight into the physics of what is happening when the small parameters are not so small?

Derive the equation for the slow dynamics

Knowing the slow dynamics evolves independently of the fast we can find the equations for the slow dynamics by ***projecting the solution and the equations onto the null space of the fast operator*** $N(\mathcal{L})$. Then one can derive the equation for the slow dynamics

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + P\left(\mathcal{N}(\overline{\mathbf{u}}, \overline{\mathbf{u}})\right) = 0$$

$$\overline{\mathbf{u}}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \in N(\mathcal{L})$$

Separation of Time Scales and slow equations

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Strong Stratification	Quasi Geostrophy	Strong Rotation
$Ro = O(1)$	$Ro \to 0$	$Ro \to 0$
$Fr \to 0$	$Fr \to 0$	$Fr = O(1)$
$Fr/Ro = f/N \to 0$	$Fr/Ro = f/N = \text{finite}$	$Fr/Ro = f/N \to \infty$
$\omega(\mathbf{k}) = \pm \frac{\lvert\mathbf{k_H}\rvert}{\lvert\mathbf{k}\rvert}$	$\omega(\mathbf{k}) = \pm \frac{(Fr^2 m^2 + Ro^2 \lvert\mathbf{k_H}\rvert)^{1/2}}{RoFr\lvert\mathbf{k}\rvert}$	$\omega(\mathbf{k}) = \pm \frac{\lvert m\rvert}{\lvert\mathbf{k}\rvert}$
$\omega(\mathbf{k}) = 0 \quad \text{(double)}$	$\omega(\mathbf{k}) = 0 \quad \text{(double)}$	$\omega(\mathbf{k}) = 0 \quad \text{(double)}$

Two kinds of frequencies: 1) zero frequency for all **k** which contribute to the potential vorticity and 2) dispersive waves with zero pv

Projection operators for the 3 limits

Strong Rotation $Ro \to 0 \;\; Fr = O(1)$

$$P_{Ro}\mathbf{u} = \begin{pmatrix} \langle \mathbf{v}_H \rangle_z - \nabla_H \Delta_H^{-1} \left(\nabla_H \cdot \langle \mathbf{v}_H \rangle_z \right) \\ \langle w \rangle_z \\ \rho \end{pmatrix}$$

Strong Stratification $Fr \to 0 \;\; Ro = O(1)$

$$P_{Fr}\mathbf{u} = \begin{pmatrix} \mathbf{v}_H - \nabla_H \Delta_H^{-1} \left(\nabla_H \cdot \mathbf{v}_H \right) \\ 0 \\ \langle \rho \rangle_H \end{pmatrix}$$

Quasi Geostrophy $Ro \to 0 \quad Fr \to 0 \quad Fr/Ro = f/N = \text{finite}$

$$P_{QG}\mathbf{u} = \begin{pmatrix} \mathbf{v}_H - Bu^2 \Delta_{QG}^{-1} \frac{\partial^2 \mathbf{v}_H}{\partial z^2} - \Delta_{QG}^{-1} \left(\nabla_H \left(\nabla_H \cdot \mathbf{v}_H \right) + Bu \nabla_H \times \left(\hat{z} \frac{\partial \rho}{\partial z} \right) \right) \\ 0 \\ \rho - Bu \Delta_{QG}^{-1} \left(\frac{\partial}{\partial z} \left(\nabla_H \times \mathbf{v}_H \right) \right) - \Delta_{QG}^{-1} \Delta_H \rho \end{pmatrix}$$

The new slow equations challenge our ideas of fast and slow dynamics in the ocean. They are *nonhydrostatic.*

$$\frac{\partial \mathbf{v}_H}{\partial t} + \mathbf{v}_H \cdot \nabla_{\mathbf{H}} \mathbf{v}_H + \nabla_{\mathbf{H}} p = \frac{1}{Re} \Delta_H \mathbf{v}_H$$ **2-D**

$$\frac{\partial w}{\partial t} + \mathbf{v}_H \cdot \nabla_{\mathbf{H}} w = \frac{1}{Re} \Delta_H w - \frac{1}{Fr} \langle \rho \rangle_z$$ **2-D**

$$\frac{D\rho}{Dt} - \frac{1}{Fr} w = \frac{1}{Re} \frac{1}{Pr} \Delta \rho$$ **3-D**

$$\nabla_{\mathbf{H}} \cdot \mathbf{v}_H = 0$$ **2-D**

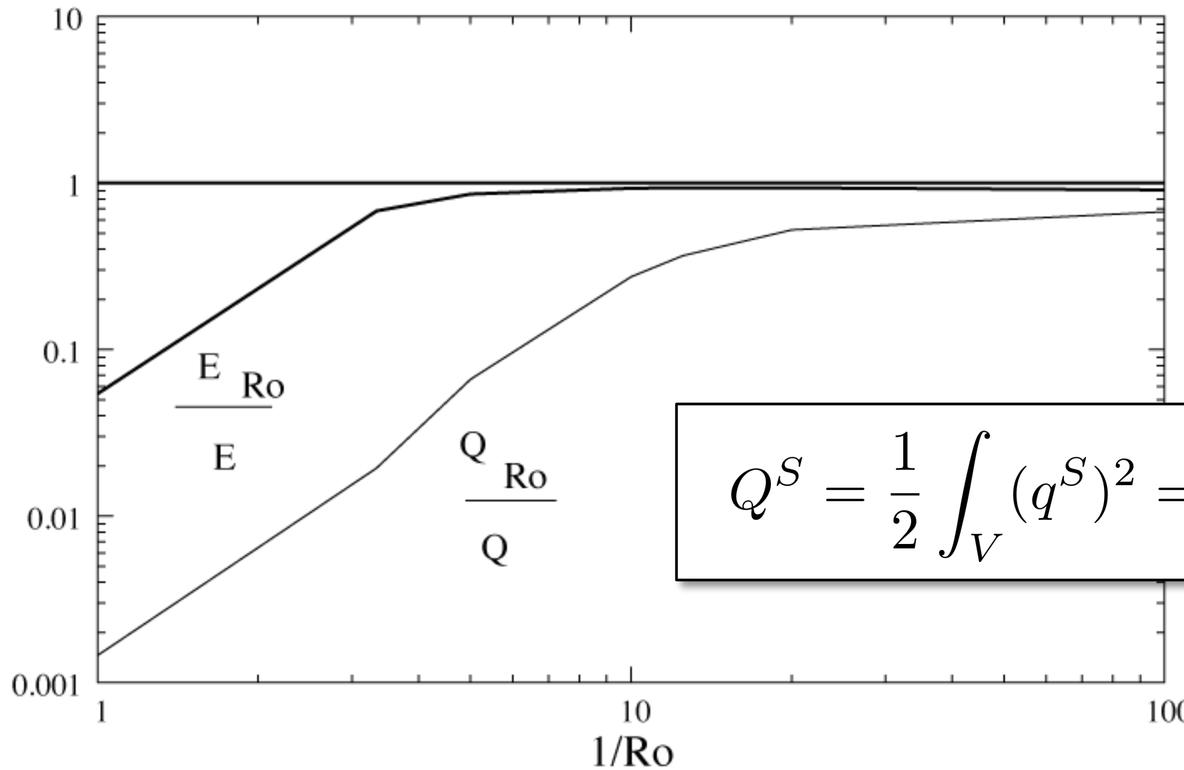
$$\mathbf{u}^S(\mathbf{x}, 0) = \mathbf{u}_0^S(\mathbf{x}) \in \mathcal{N}(L_F)$$

$$\mathbf{v} = \mathbf{v}(x, y, t), \rho = \rho(x, y, z, t), \text{ and } \nabla_H \, p = \nabla_H \Delta_H^{-1} \Big(\nabla_H \cdot (\mathbf{v}_H \cdot \nabla_H \mathbf{v}_H) \Big)$$

"Low Rossby limiting dynamics for stably stratified flows", B. Wingate, P. Embid, M. Holmes-Cerfon, M. Taylor, Journal of Fluid Mechanics, 2011

Also found nonhydrostatic slow equations:
"Generalized Quasi-Geostrophy for spatially anisotropic rotationally constrained flows", K. Julien, E. Knoblocch R. Millif, J. Werne, Journal of Fluid Mechanics, 2006

Dependence on Rotation Rate – wave number 3 white noise forcing – triply periodic box



$$Q^S = \frac{1}{2}\int_V (q^S)^2 = \frac{1}{2}\int_V (\rho_z)^2$$

$$Q = \frac{1}{2}\int_V q^2$$

$$q = \frac{\partial \rho}{\partial z} - \frac{Ro}{Fr}\omega_3 + Ro\,(\omega \cdot \nabla\rho)$$

Rewrite the full Boussinesq Equations (three times) using

$$\begin{aligned} \mathbf{u} &= \mathbf{u}^{\alpha} + \mathbf{u}'_{\alpha} \\ P_{\alpha}\mathbf{u}^{\alpha} &= \mathbf{u}^{\alpha} \\ P_{\alpha}\mathbf{u}' &= 0 \end{aligned}$$

Strong Rotation $Ro \to 0 \;\; Fr = O(1)$

$$\frac{\partial \mathbf{v}_H^{Ro}}{\partial t} + \mathbf{v}_H^{Ro} \cdot \nabla_H \mathbf{v}_H^{Ro} - \nabla_H \Delta_H^{-1} \left(\nabla_H \cdot \left(\mathbf{v}_H^{Ro} \cdot \nabla_H \mathbf{v}_H^{Ro} \right) \right) - \frac{1}{Re} \Delta_H \mathbf{v}_H^{Ro}$$

$$= - \left\langle \left(1 - \nabla_H \Delta_H^{-1} \nabla_H \cdot \right) \langle \mathbf{v}' \cdot \nabla \mathbf{v}' \rangle_z \right\rangle_H ,$$

$$\nabla_H \cdot \mathbf{v}_H^{Ro} = 0,$$

$$\frac{\partial w^{Ro}}{\partial t} + \mathbf{v}_H^{Ro} \cdot \nabla_H w^{Ro} + \frac{1}{Fr} \langle \rho \rangle_z - \frac{1}{Re} \Delta_H w^{Ro} = - \langle \mathbf{v}' \cdot \nabla w' \rangle_z ,$$

$$\frac{\partial \rho}{\partial t} + \mathbf{v}^{Ro} \cdot \nabla \rho - \frac{1}{Fr} w^{Ro} - \frac{1}{RePr} \Delta \rho = - \mathbf{v}' \cdot \nabla \rho + \frac{1}{Fr} w' .$$

For strong rotation we get the following volume averaged energy equation:

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}_H^{Ro}\|_2^2 + \frac{1}{Re}\|\nabla_H \boldsymbol{v}_H^{Ro}\|_2^2 = -\int_A \boldsymbol{v}_H^{Ro} \cdot \langle\{\boldsymbol{v}' \cdot \nabla \boldsymbol{v}'\}_H\rangle_z dA, \quad (1)$$

$$\frac{1}{2}\frac{d}{dt}\|w^{Ro}\|_2^2 + \frac{1}{Re}\|\nabla_H w^{Ro}\|_2^2 = -\frac{1}{Fr}\int_A w^{Ro}\langle\rho\rangle_z dA - \int_A w^{Ro}\langle \boldsymbol{v}_H' \cdot \nabla_H w'\rangle_z dA, \quad (2)$$

$$\frac{1}{2}\frac{d}{dt}\|\rho\|_2^2 + \frac{1}{Re\,\mathrm{Pr}}\|\nabla\rho\|_2^2 = \frac{1}{Fr}\int_V w^{Ro}\rho dV + \frac{1}{Fr}\int_V w'\rho dV. \quad (3)$$

...And for QG

$$\frac{1}{2}\frac{d}{dt}\left\|\boldsymbol{v}_H^{QG}\right\|_2^2 - \frac{1}{Re}\|\nabla\boldsymbol{v}_H^{QG}\|_2^2 = \boxed{-Bu\int \boldsymbol{v}_H^{QG}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\left[\boldsymbol{v}_H^{QG}\cdot\nabla\rho^{QG}\right]\right)\right]} \tag{1}$$

$$-\int \boldsymbol{v}_H^{QG}\cdot\left(1-Bu^2\Delta_{QG}^{-1}\frac{\partial^2}{\partial z^2}\right)\{\boldsymbol{v}'\cdot\nabla\boldsymbol{v}'\}_H - Bu\int \boldsymbol{v}_H^{QG}\cdot\Delta_{QG}^{-1}\left[\nabla_H\times\left(\hat{z}\frac{\partial}{\partial z}\{\boldsymbol{v}\cdot\nabla\rho'\}\right)\right]$$

$$\frac{1}{2}\frac{d}{dt}\|\rho^{QG}\|_2^2 - \frac{1}{Re\,\mathrm{Pr}}\|\nabla\rho\|_2^2 = \boxed{Bu\int \rho^{QG}\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(\boldsymbol{v}_H^{QG}\cdot\nabla_H\omega^{QG}\right)} \tag{2}$$

$$-\int \rho^{QG}\left(1-\Delta_{QG}^{-1}\Delta_H\right)\boldsymbol{v}'\cdot\nabla\rho' + Bu\int \rho^{QG}\Delta_{QG}^{-1}\frac{\partial}{\partial z}\left(\nabla_H\times\{\boldsymbol{v}\cdot\nabla\boldsymbol{v}'\}_H\right).$$

Numerical simulations - 512^3

$$L_d = \frac{N}{f} L_f \qquad \text{or} \qquad k_d = \frac{f}{N} k_f$$

Gaussian forcing spectrum:

$$F(k) = \epsilon_f \frac{\exp(-.5(k - k_f)^2/s^2)}{(2\pi)^{1/2} s}$$

Fast rotation limit (the flow makes columns)

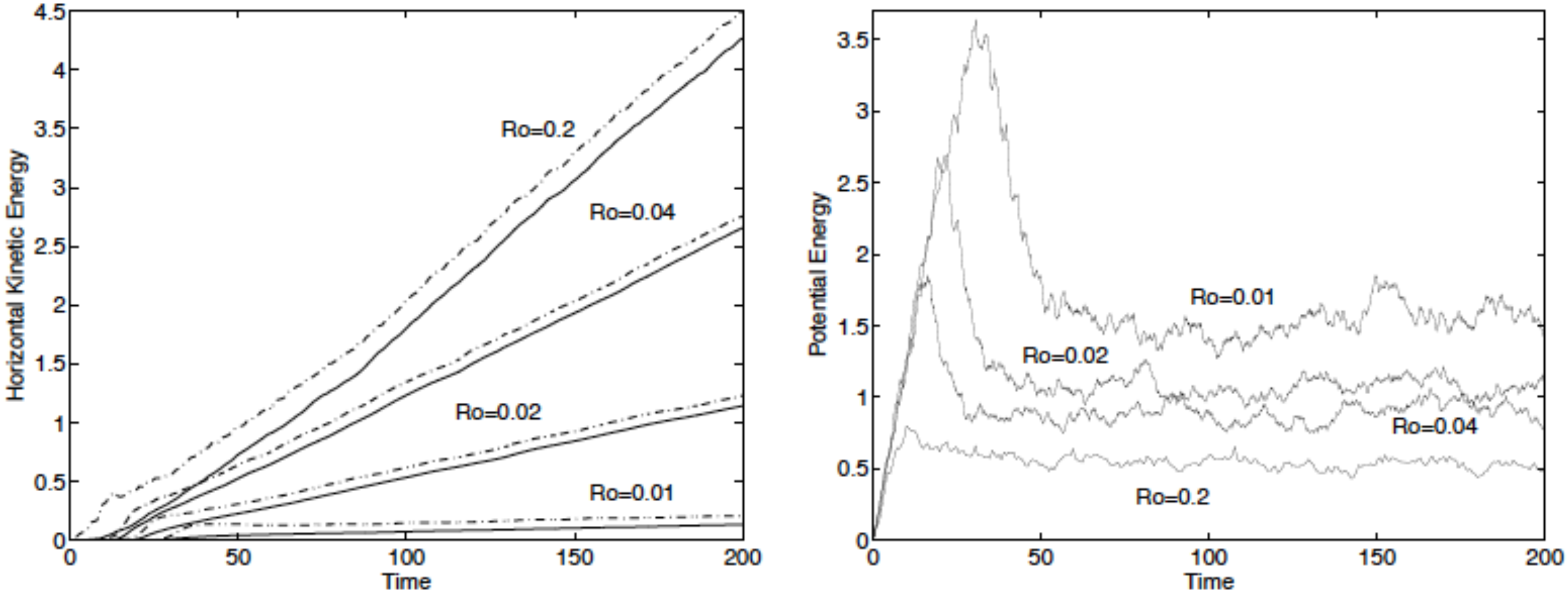


FIGURE 1. The evolution of the horizontal kinetic energy (left) and potential energy (right) for a handful of the rapidly rotating and weakly stratified simulations. The dashed lines in the left plot indicate the full horizontal kinetic energy while the solid lines represent the development of the horizontal kinetic energy on the slow manifold.

Fast rotation limit (the flow makes columns)

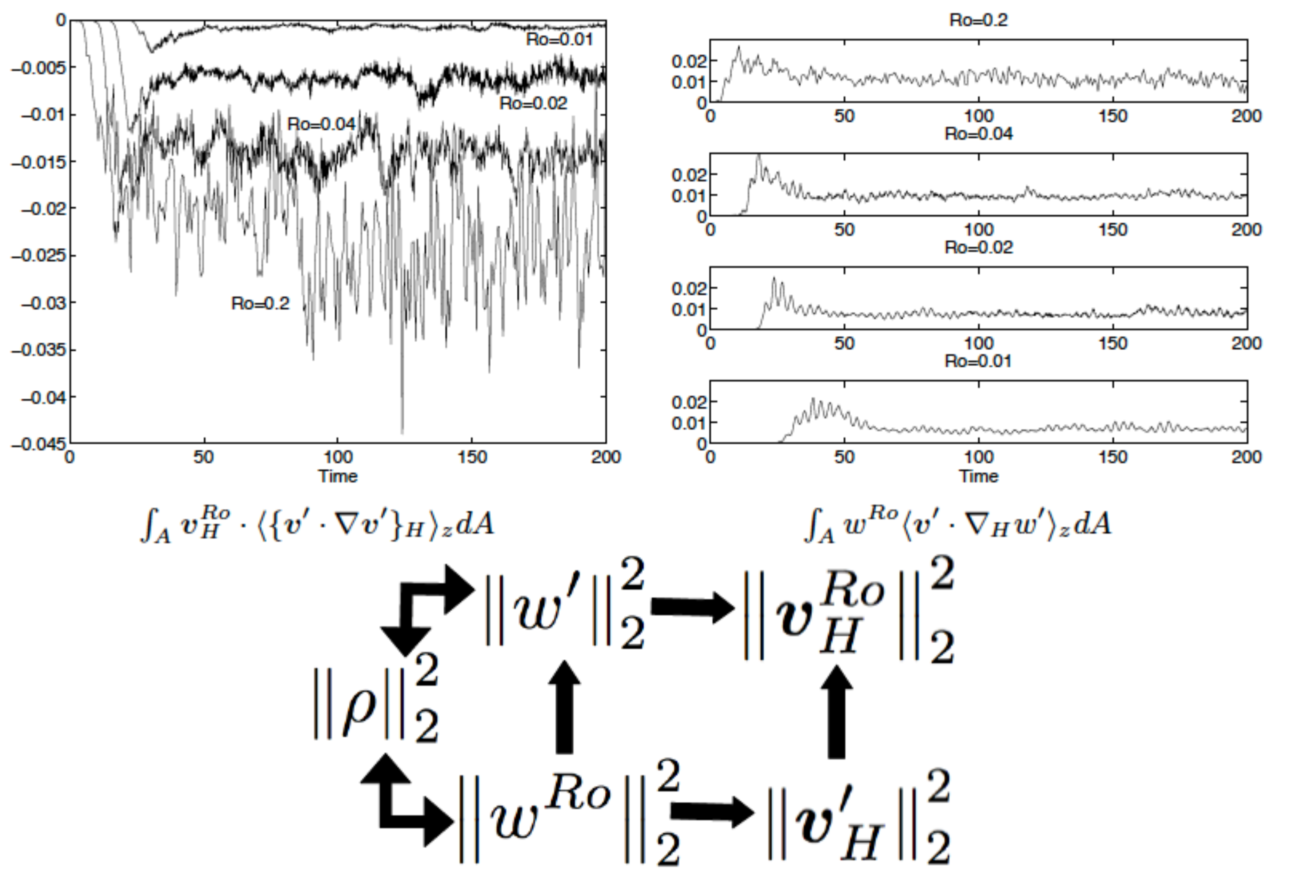


FIGURE 2. The exchange between the non-slow components of the flow and the slow horizontal kinetic energy (left) and slow vertical kinetic energy (right) for a handful of the rapidly rotating, weakly stratified simulations. Each of these quantities has a definite sign (for all time t) for $Ro \leqslant 2$. A schematic of the movement of the energy is also shown.

Strong stratification limit (the flow makes sheets)

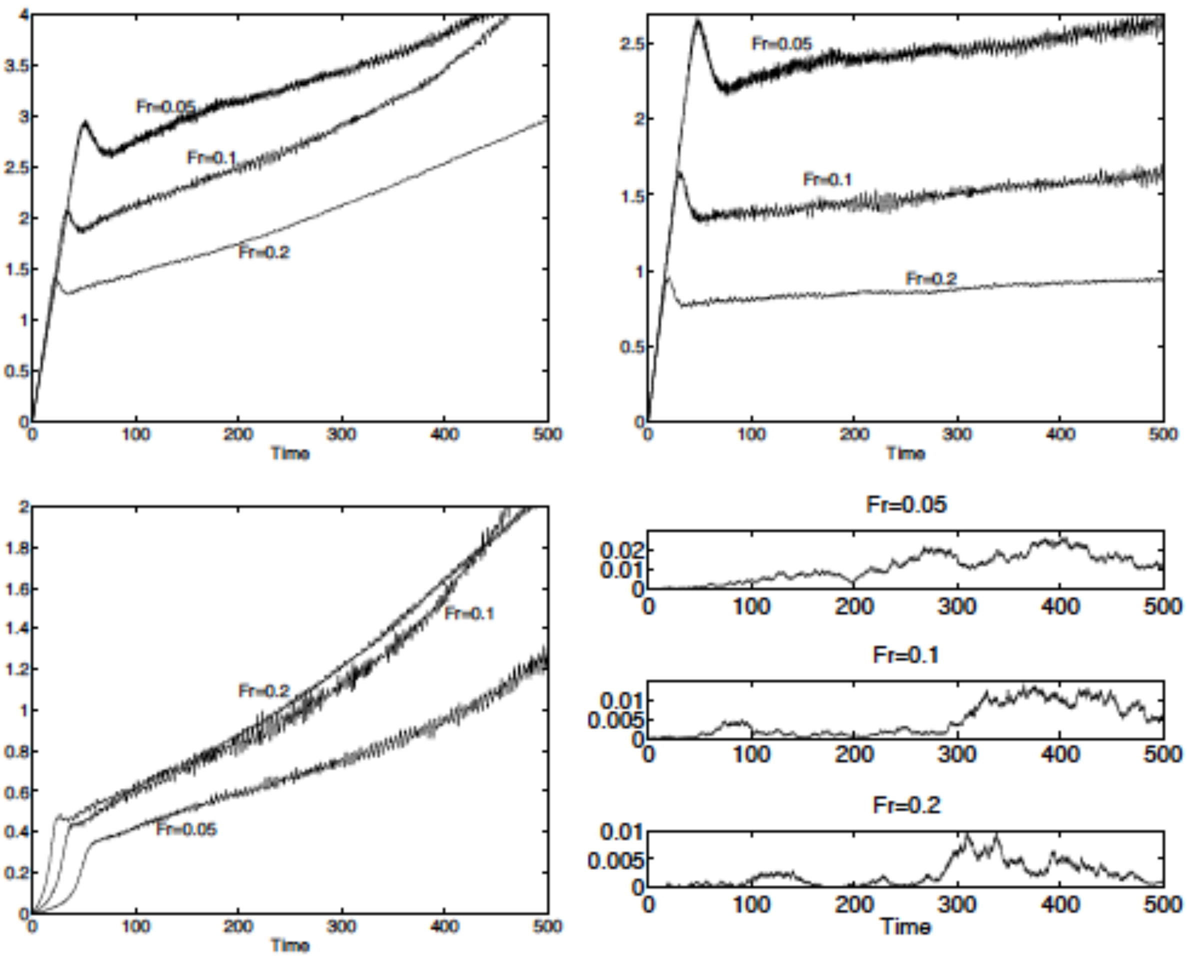
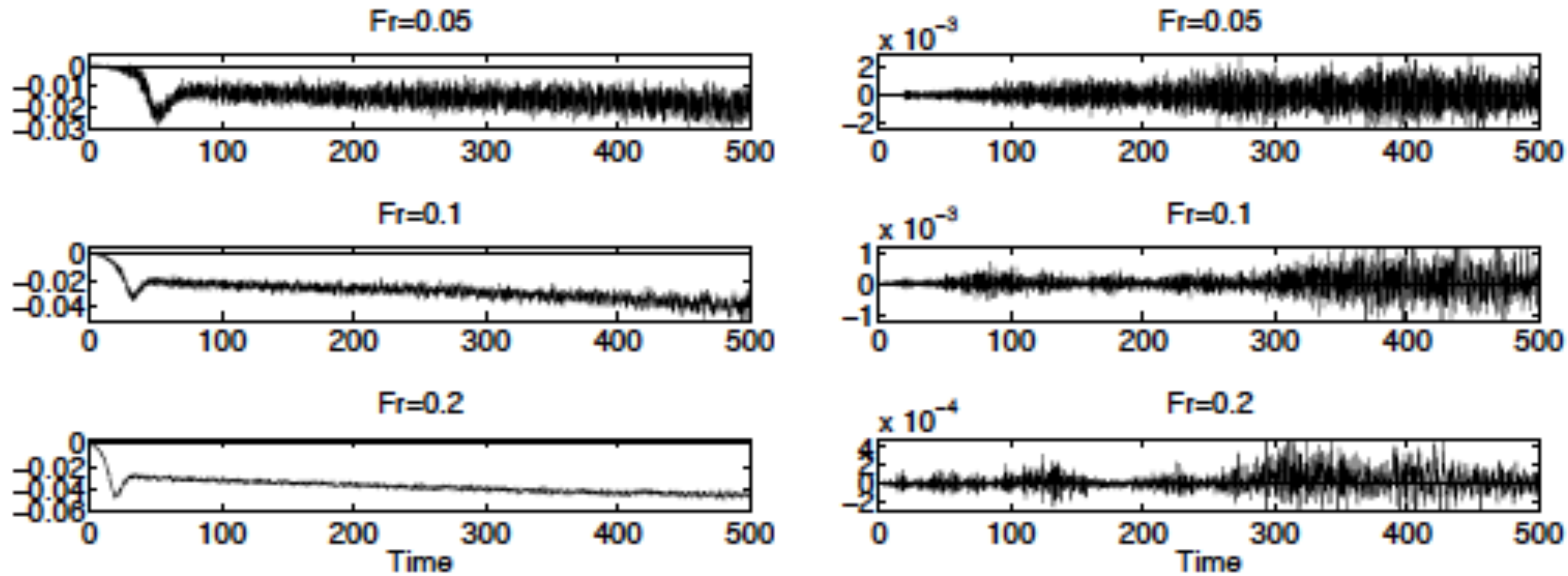


FIGURE 3. The evolution of the kinetic (left) and potential (right) energy. The upper plots are the full energy of the system ($\boldsymbol{u} = \boldsymbol{u}^{Fr} + \boldsymbol{u}'$) and the lower plots are the energy projected onto the slow manifold.

Strong stratification limit (the flow makes columns)



$\int_V v_H^{Fr} \cdot \{v' \cdot \nabla v'\}_H dV$ $\int \rho^{Fr} \langle v' \cdot \nabla \rho' \rangle_H dz$

FIGURE 4. The exchange of energy between the non-slow parts of the flow and the kinetic (left) and potential (right) energy on the slow manifold. While the non-slow momentum drives the momentum on the slow manifold, the influence of the density is less clear, i.e. the right hand plots do not exhibit a definitive exchange.

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The QG limit –dashed is the full system, solid is slow

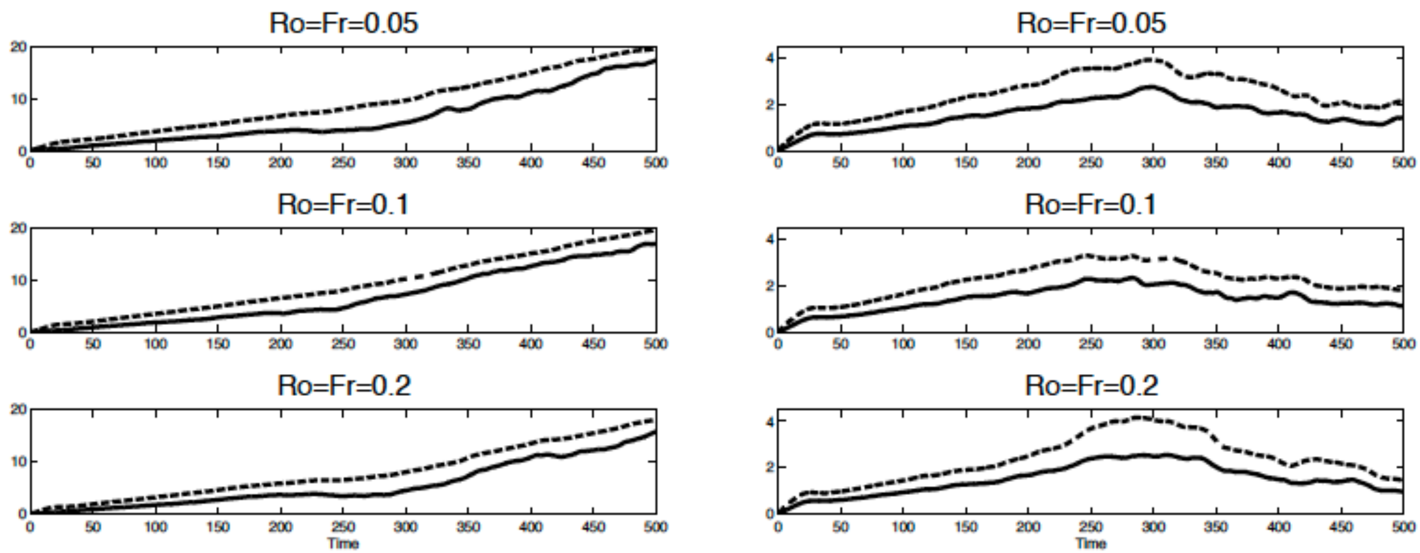
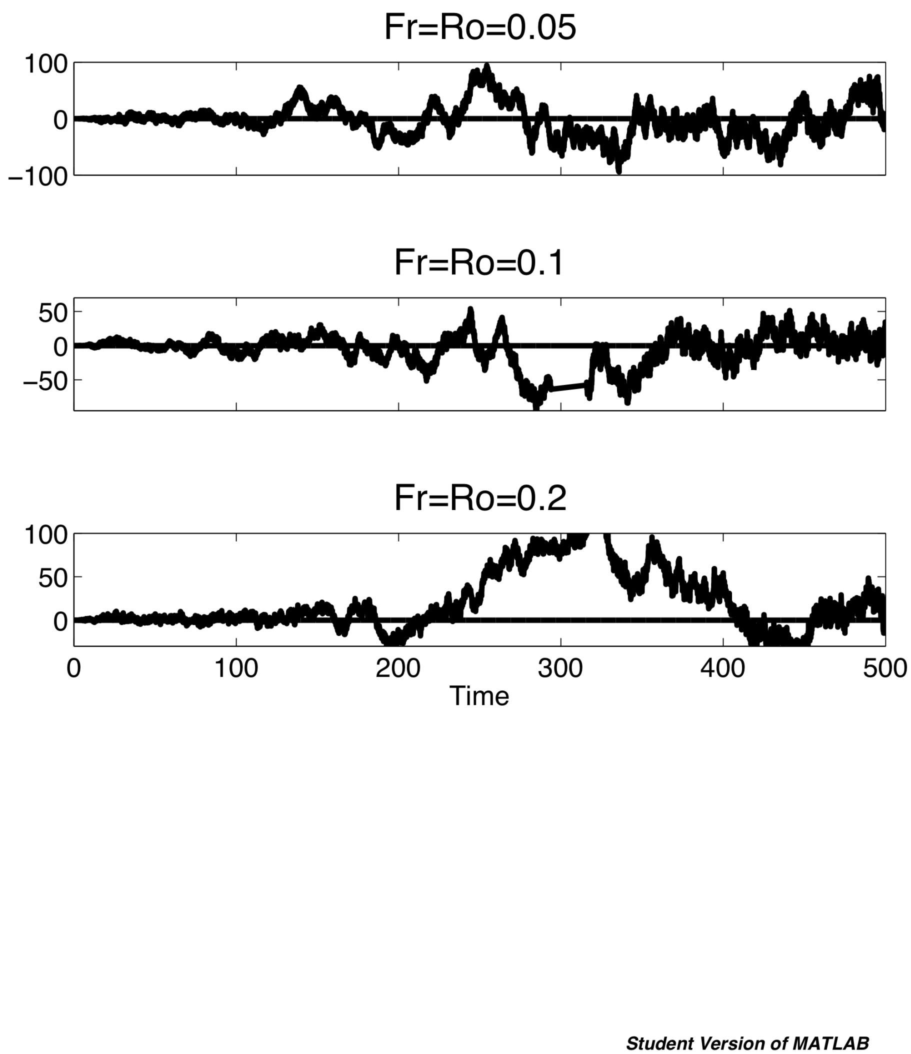


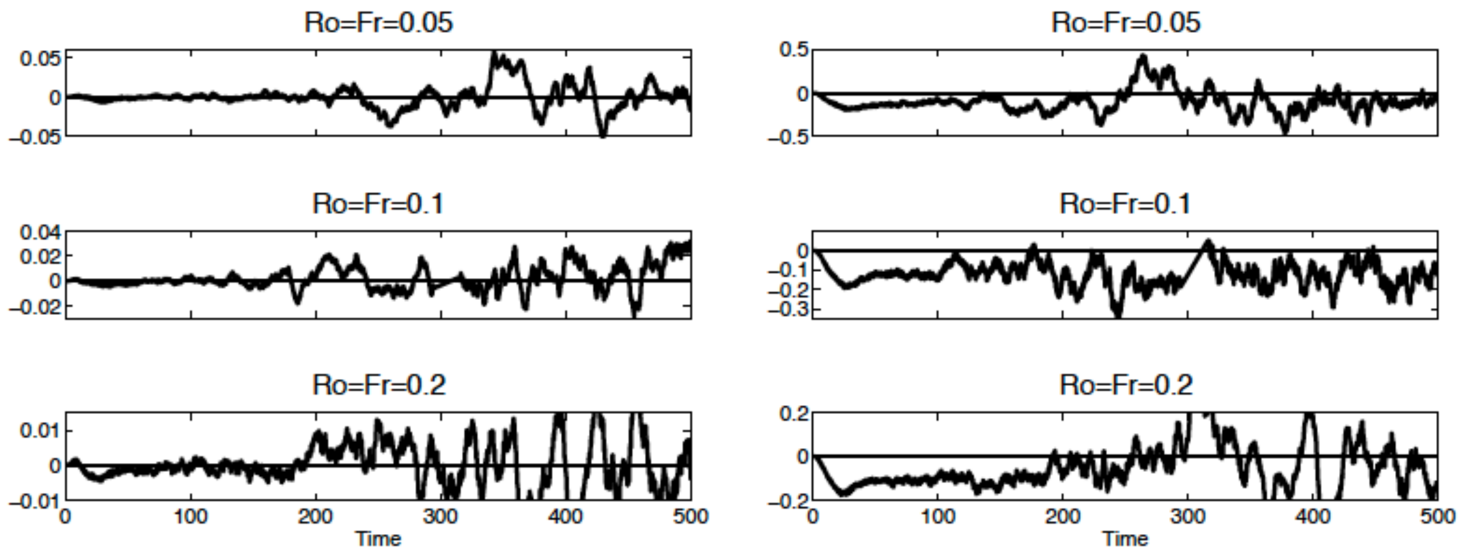
FIGURE 5. The evolution of the kinetic energy (left) and potential energy (right) for a handful of the rapidly rotating and strongly stratified simulations (quasi-geostrophy). The dashed lines indicate the energy of the full system, while the solid lines is the energy on the slow manifold.

Despite the fact that the forcing is directly on the buoyancy equation it is the slow horizontal kinetic energy that continues to grow, the potential energy reaches a maximum half way through the simulation.

QG limit continued – slow PE to slow KE



QG limit continued – fluctuations on the slow kinetic energy

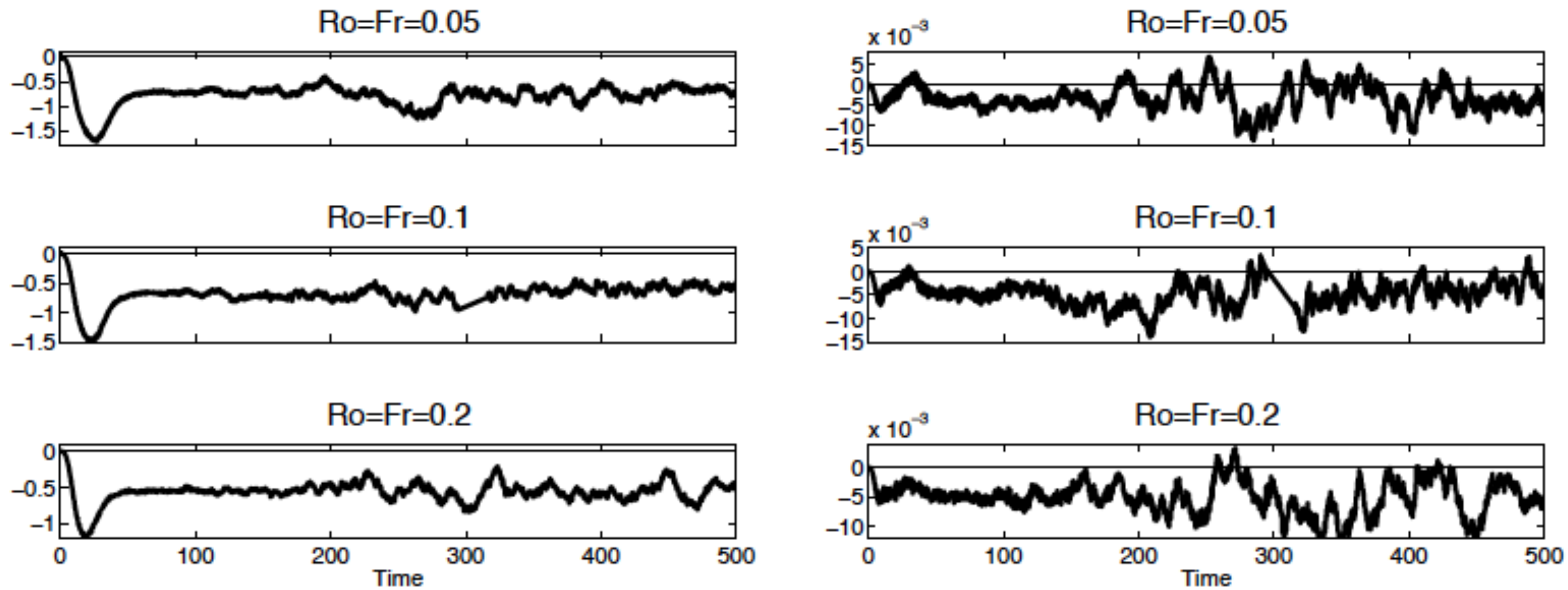


$$\int v_H^{QG} \cdot \left(1 - Bu^2 \Delta_{QG}^{-1} \frac{\partial^2}{\partial z^2}\right) \{v' \cdot \nabla v'\}_H \qquad Bu \int v_H^{QG} \cdot \Delta_{QG}^{-1} \left[\nabla_H \times (\hat{z}\{v' \cdot \nabla \rho'\}_z)\right]$$

FIGURE 6. The transfer of energy from the fluctuating components of the flow to the slow kinetic energy.

The conversion of slow potential to kinetic is larger by far than any of these terms in the equations. And as the parameters get smaller it makes more sense to study the evolution of the potential enstrophy and it's fluctuations.

Transfer of energy from the fluctuation components of the flow to the slow potential energy



$$Bu \int \rho^{QG} \Delta_{QG}^{-1} \frac{\partial}{\partial z} \left(\nabla_H \times \{ \boldsymbol{v}' \cdot \nabla \boldsymbol{v}' \}_H \right) \qquad \int \rho^{QG} \left(1 - \Delta_{QG}^{-1} \Delta_H \right) \{ \boldsymbol{v}' \cdot \nabla \rho' \}$$

FIGURE 7. The transfer of energy from the fluctuating components of the flow to the slow potential energy.

Summary

- For all three limits the non-slow part of the flow served to move energy to and from the slow manifold.
- For Bu = 1 Quasi-Geostrophic flows
  1. most of the transfer was accomplished via slow-to-slow transfers
  2. the non-slow part of the buoyancy dynamics was still responsible for moving energy from potential to slow-kinetic
  3. The non-slow momentum acts as a sink for the slow potential energy
  4. The slow potential energy not only get converted to slow kinetic but can also shed off of the slow part of the dynamics as non-slow kinetic energy
- What about potential enstrophy transfers?