Mesoscale tropospheric motions as a three time scale problem: towards a rigorous justification of the pseudo-incompressible model

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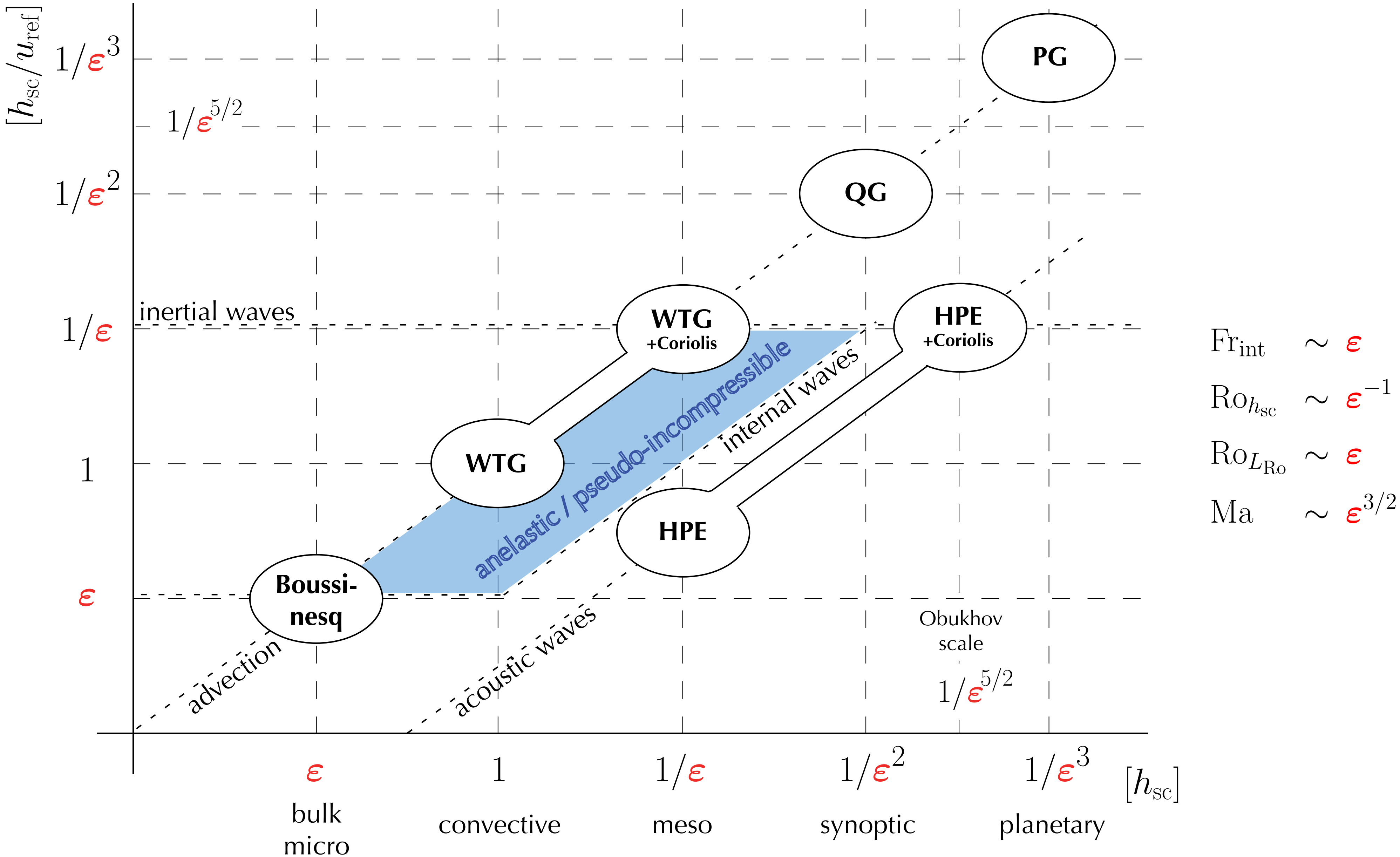
Background on sound-proof models

Formal asymptotic regime of validity

Steps towards a rigorous proof

Summary

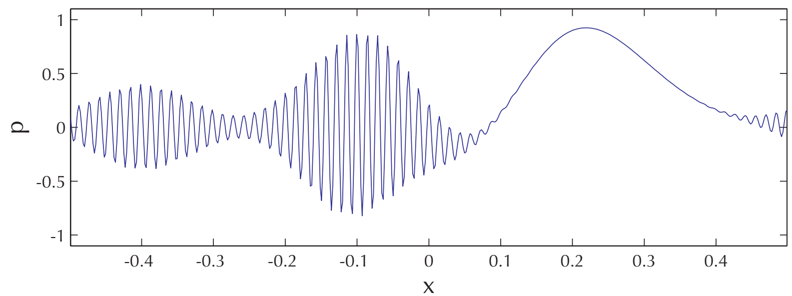
Atmospheric Flow Regimes



R.K., Ann. Rev. Fluid Mech, **42**, 2010

Motivation ... Numerics

Why not simply solve the full compressible equations?



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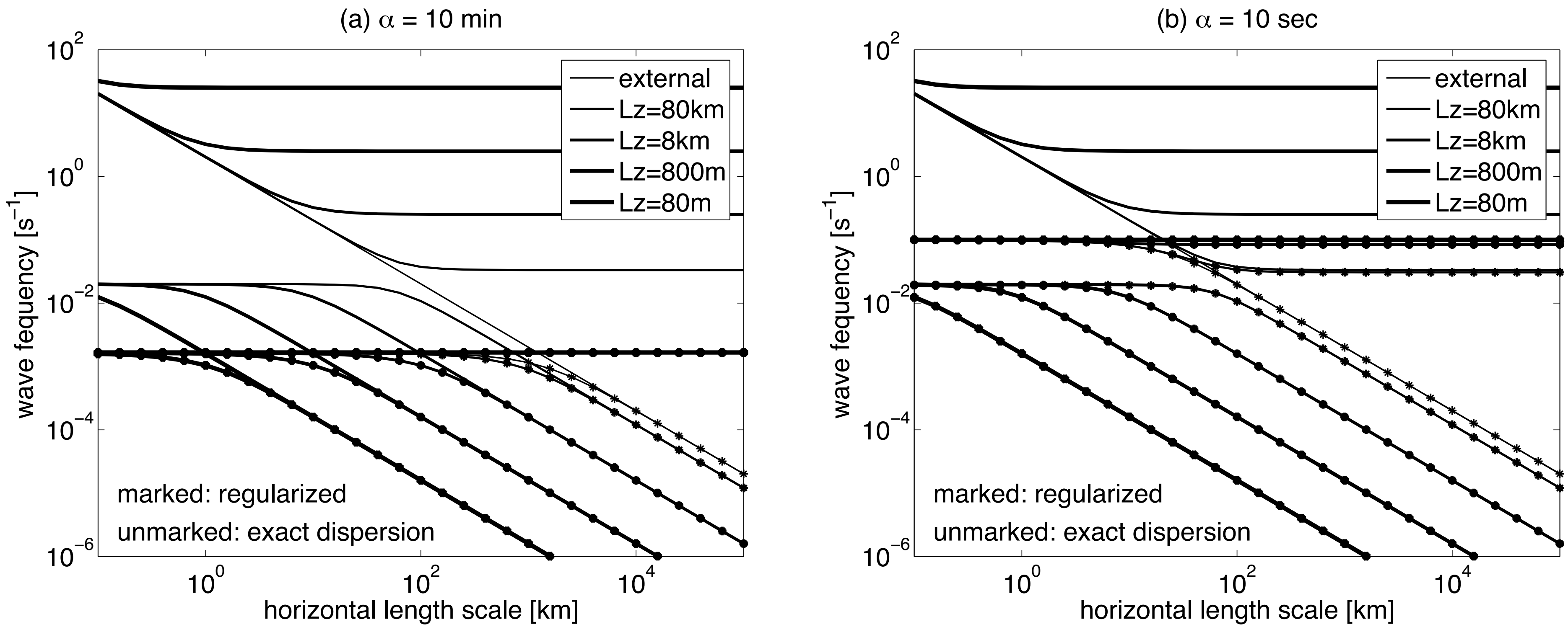


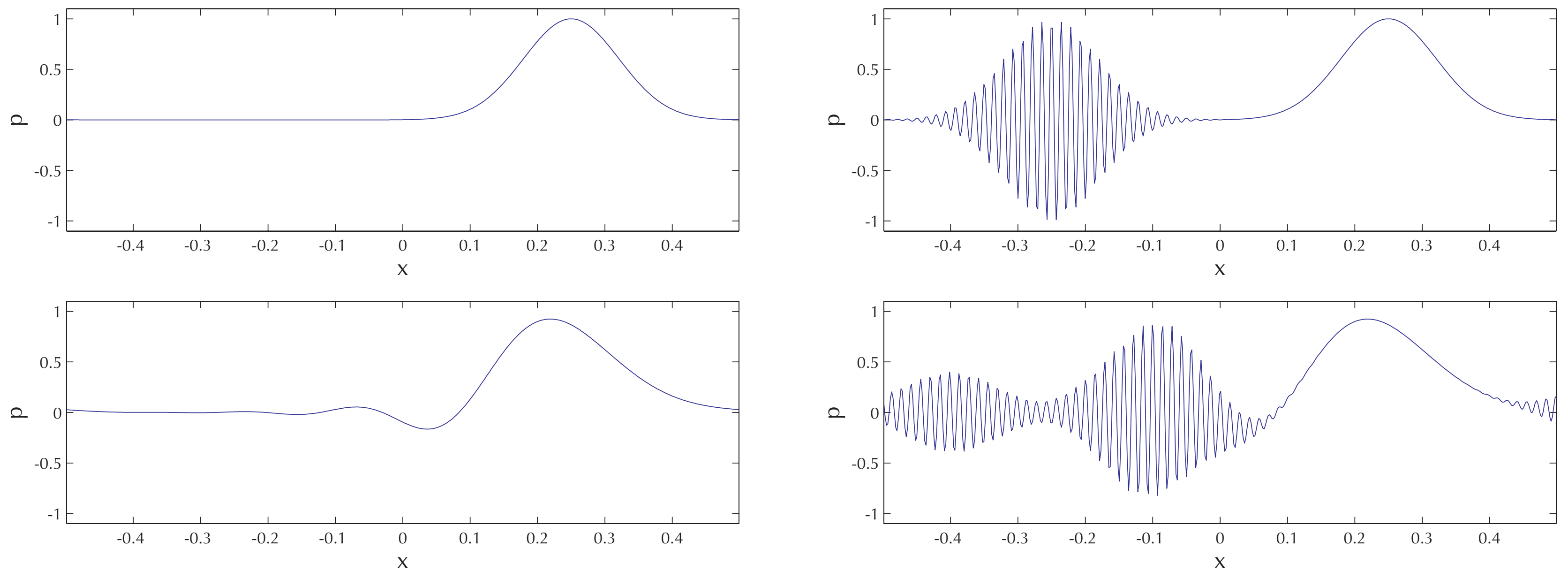
* adapted from Reich et al. (2007)

Motivation ... Numerics

Why not simply solve the full compressible equations?

Linear Acoustics, simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



$t = 0$

$t = 3$

Central question:

What is the correct small-scale behavior
of a
full compressible flow solver
for sub-acoustic time scales

Sound-Proof Models

Compressible flow equations

$$
\begin{aligned}
\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
\boldsymbol{(\rho w)_t} \; \boldsymbol{+ \nabla \cdot (\rho v w)} + P \pi_z &= -\rho g \\
\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

drop term for:

anelastic† (approx.)

pseudo-incompressible*

hydrostatic-primitive

$$
P = p^{\frac{1}{\gamma}} = \rho\theta \,, \qquad \pi = p/\Gamma P \,, \qquad \Gamma = c_p/R \,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Parameter range & length and time scales of asymptotic validity ?

† e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

* Durran, JAS, **46**, 1453–1461 (1989)

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From here on: ε is the Mach number

Regimes of Validity ... Design Regime

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

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Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips (1962)

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Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\mathrm{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \Rightarrow \qquad \Delta\overline{\theta}\,\Big|_{z=0}^{h_{\mathrm{sc}}} < 1\ \mathrm{K}$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu} \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Regimes of Validity ... Design Regime

Full compressible flow equations in perturbation variables

$$
\begin{aligned}
\tilde{\theta}_\tau &+ \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} && && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau &- \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} && + \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \ \cdot \\
\tilde{\pi}_\tau & && + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

For the linear variable coefficient system:

✓ Conservation of weighted quadratic energy

✓ Control of time derivatives by initial data $(\tau = O(1))$

✓ Control of horizontal derivatives

✓ Control of vertical derivatives via eigenmode expansions (see below)

(•) Control of nonlinear resonances (see below)

Regimes of Validity ... Design Regime

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_{\vartheta} \; + \; \tilde{w}\,\frac{d\overline{\theta}}{dz} \qquad = \; 0$$

$$\tilde{\boldsymbol{v}}_{\vartheta} \; - \; \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^{*} \qquad = \; 0$$

$$\boldsymbol{\varepsilon}^{\boldsymbol{\mu}}\,\pi^{*}_{\vartheta} + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) \; = \; 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^{*} \end{pmatrix} (\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^{*} \\ \boldsymbol{U}^{*} \\ W^{*} \\ \Pi^{*} \end{pmatrix} (z) \;\; \exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Regimes of Validity ... Design Regime

Relation between compressible and pseudo-incompressible vertical modes

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\overline{\theta}\,\overline{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}\,\overline{P}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}\,\overline{P}}W^*$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes *(rigid lid)*

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem* ...

$$\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1) \;:$$ perturbations of pseudo-incompressible modes & EVals

*** Taylor-Goldstein equation**

Regimes of Validity ... Design Regime

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\bar{\theta}\,\bar{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\bar{\theta}\,\bar{P}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\bar{\theta}\,\bar{P}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)\right)$

- pseudo-incompressible modes/EVals = compressible modes/EVals + $O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

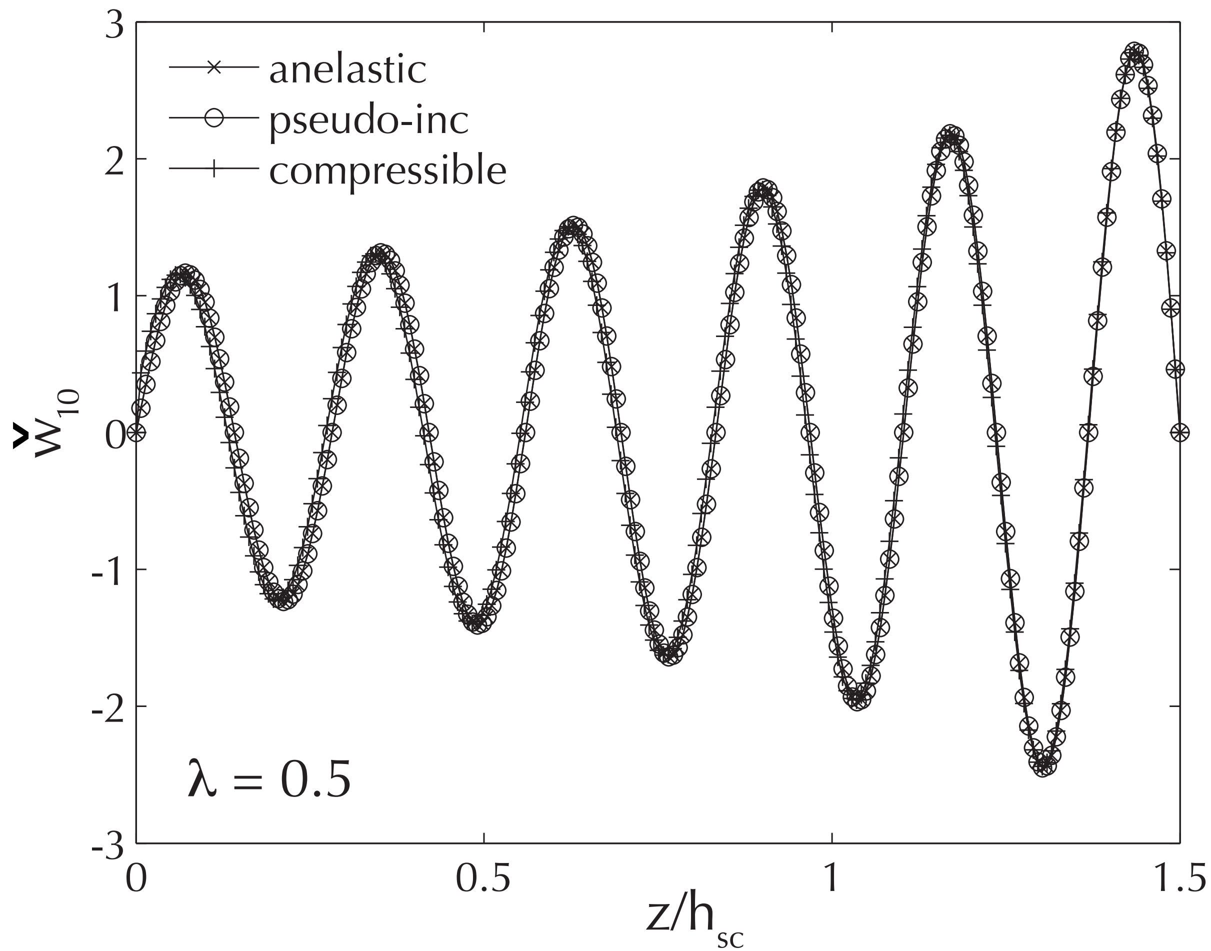
$$\frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{\rm sc}} \lesssim 40 \text{ K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

† rigorous proof with D. Bresch

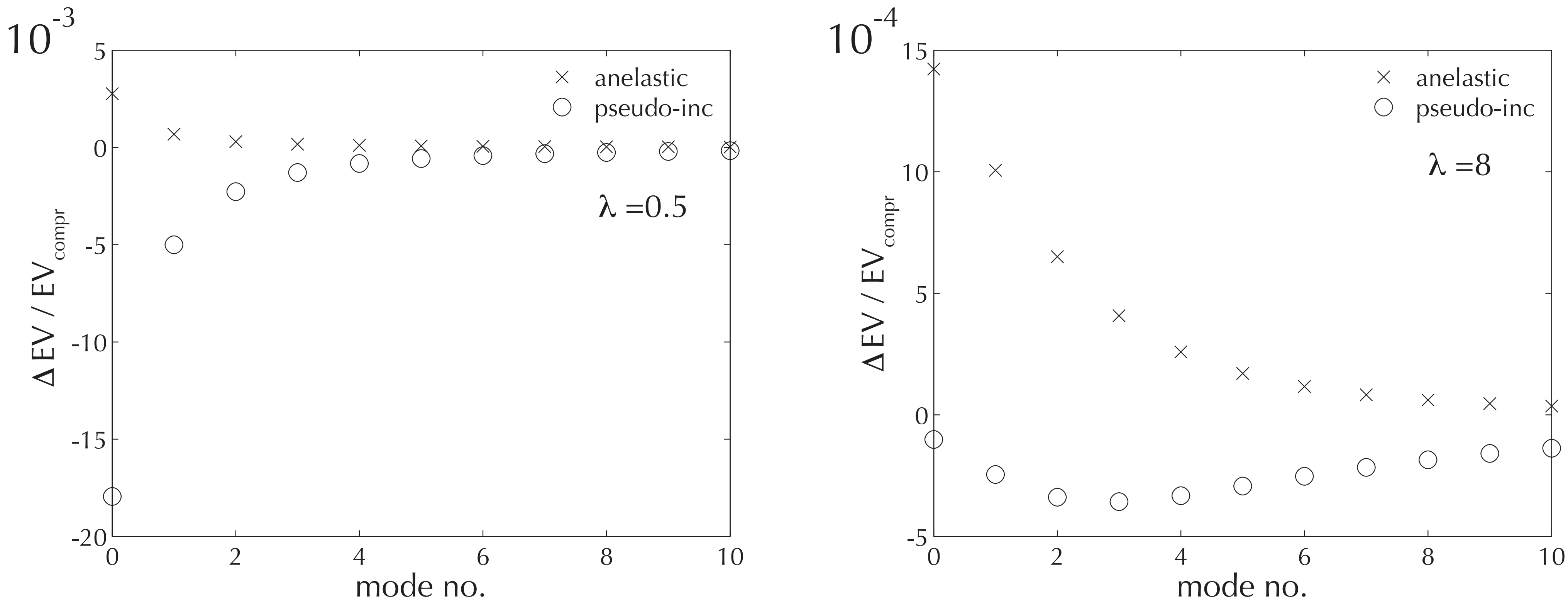
Regimes of Validity ... Design Regime

A typical vertical structure function $(L \sim \pi h_{sc} \sim 30\ \mathrm{km};\quad \boldsymbol{\varepsilon^{\mu} = 0.1})$



thanks to Dr. V. LeDoux, Ghent, for the SL-solver MATSLISE!

Relative errors for the eigenvalues $(\varepsilon^{\mu} = 0.1)$



$$\frac{EV_{sproof} - EV_{compr}}{EV_{compr}}$$

thanks to Dr. V. LeDoux, Ghent, for the SL-solver MATSLISE!

Regimes of Validity ... Design Regime

Sample **EULAG**-simulation of a $\lambda = 106\ \mathrm{km}, m = 0$ – eigenmode for

$$\overline{\theta}(z) = \frac{T_{\mathrm{ref}}}{1 - 0.1(z/h_{\mathrm{sc}})}$$

Thanks to P. Smolarkiewicz

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Steps in the proof

$$
\begin{aligned}
\tilde{\theta}_\tau + \frac{1}{\varepsilon^\nu}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} \qquad\qquad &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau + \frac{1}{\varepsilon^\nu}\,\frac{\tilde{\theta}}{\overline{\overline{\theta}}}\,\boldsymbol{k} + \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi}\ \cdot \\
\tilde{\pi}_\tau + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) &= -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

Existence & uniqueness of solutions for $t \leq T$ with T independent of ε

1. via energy estimates*
   - L^2 **control of derivatives in the fast linear system**
   - nonlinear terms: Picard iteration exploiting Sobolev embedding
2. via spectral expansions (on bounded domains)*
   - "non-resonance" through non-linear terms or
   - effective eqs. for resonant subsets of modes

*Majda, Metivier, Schochet, Embid, ...

*Babin, Mahalov, Nicolaenco, Dutrifoy ...