Kinetic-Induced Moment System for Conservation Laws

Universität Hamburg
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Project Aims and Challenges

This project focuses in the derivation of **Kinetic-Induced Moment Systems (KIMS)** based in the relationship between kinetic theory and non-linear hyperbolic conservation laws and their potential applications in the description of small-scale geophysical flows. Using simple base cases as the 1-D Burgers' equation and the 1-D shallow water equations, our aim on the one hand is to use the resulting PDEs as a monitoring function to detect particular flow structure (like shocks and rarefaction waves) into the construction of adaptive numerical methods, and on the other hand as a basis to derive novel parametrizations for subgrid closures.

Main Definitions

- **Boltzmann-like equation with a BGK collision term:** describes the statistical distribution of the density of particles $f(t,x,\xi)$, where $f_0(Q,\xi)$ is considered its *equilibrium function* and ξ the microscopic velocity,

$$\partial_t f(t,x,\xi) + \xi\partial_x f(t,x,\xi) = \frac{1}{\varepsilon}[f_0(Q,\xi) - f(t,x,\xi)]$$

with $0 < \varepsilon \ll 1$, the mean free path.

- **Moments (W_k):** are weighted averages of $f(t,x,\xi)$,

$$W_k = \int_{\mathbb{R}} \xi^k f(t,x,\xi)\,\mathrm{d}\xi$$

with $k \in \mathbb{Z}_{\geq 0}$ the order of the moment.

The **equilibrium moments $W_k\,|_E$** are also weighted averages but of $\boldsymbol{f_0(Q,\xi)}$,

$$W_k\,|_E = \int_{\mathbb{R}} \xi^k f_0(Q,\xi)\,\mathrm{d}\xi$$

Consider a 1-D conservation laws system

$$\partial_t Q + \partial_x F(Q) = 0$$

$Q(n)$ the vector of unknowns with n elements. For $0 < k \leqslant n-1$ holds that $f(t,x,\xi) = f_0(Q,\xi)$ and

$$W_{k-1} = W_{k-1}\,|_E = Q(k)\,, \qquad 0 < k \leqslant n-1$$

and the remaining moments W_k for $k \geqslant n$, will yield the new unknowns.

What is a KIMS?

It is an infinite moment system based on an artificial Boltzmann-like transport equation using the **connection between kinetic theory and conservation laws** together with an asymptotic expansion of the corresponding moments.

How to derive it?

1. Multiplication on both sides of the Boltzmann-like transport equation by the weights $(1, \xi, ..., \xi^{n-1}, \xi^k)$ and subsequent integration over the microscopic velocity ξ yield an infinite PDE moment system.

2. Express the infinite system in terms of the p-order non-equilibrium moments, starting at $p = 1$

$$W_k^{(p)} = W_k^{(p-1)} - W_k^{(p-1)}\,|_E, \qquad p \geqslant 1$$

for $0 \geq k \geq n+p-2$ holds that $W_k^{(p)} = 0$.

3. Use asymptotic expansion in terms of the small parameter ε to show that at each order a scale-induced closure is possible, resulting in a closed moment system.

$$W_k^{(p)} = \varepsilon^p W_{k,p}^{(p)} + \varepsilon^{p+1} W_{k,p+1}^{(1)} + \ldots, \qquad k \geq n+p-1$$

Base Cases

1. 1-D Inviscid Burgers' equation:

$$\partial_t u + \partial_x\left(\frac{1}{2}u^2\right) = 0$$

evolution of the horizontal velocity $u(t,x)$ at a time $t \geq 0$ and at a point $x \in \mathbb{R}$.

2. 1-D Shallow Water equations:

$$\partial_t h + \partial_x(hu) = 0$$
$$\partial_t(hu) + \partial_x\left(hu^2 + \frac{g}{2}h^2\right) = 0$$

evolution of the height of water $h(t,x)$ and its horizontal velocity $u(t,x)$, with $t \geq 0$, $x \in \mathbb{R}$ and g the gravitational acceleration.

The respective KIMS at third order ($p = 3$) reads,

1-D Inviscid Burgers' equation:

$$\partial_t u + u\partial_x u + \varepsilon\partial_x W = 0$$
$$\partial_t W + \frac{1}{3\varepsilon}\partial_x u + u\partial_x W = \frac{4\varepsilon}{15}\partial_{xx} W - \frac{1}{\varepsilon}W$$

where $W = \frac{1}{\varepsilon}W_1^{(1)}$

1-D Shallow Water equations:

$$\partial_t h + \partial_x(hu) = 0$$
$$\partial_t(hu) + \partial_x(hu^2 + \frac{g}{2}h^2) + \varepsilon\partial_x W = 0$$
$$\partial_t W + \left(\frac{g}{2\varepsilon}h^2 + 3W\right)\partial_x u + u\partial_x W$$
$$= -\frac{3}{2}\varepsilon g\partial_x\left[\frac{g}{\varepsilon}h^2\partial_x h + 2W\partial_x h + h\partial_x W + gh^3\partial_{xx}u\right] - \frac{1}{\varepsilon}W$$

where $W = \frac{1}{\varepsilon}W_2^{(1)}$

Monitoring Functions ($\varepsilon \to 0$)

In the limit $\varepsilon \to 0$, the previous third-order systems read

1-D Inviscid Burgers' equation:

$$\partial_t u + u\partial_x u = 0$$
$$W = -\frac{1}{3}\partial_x u$$

1-D Shallow Water equations:

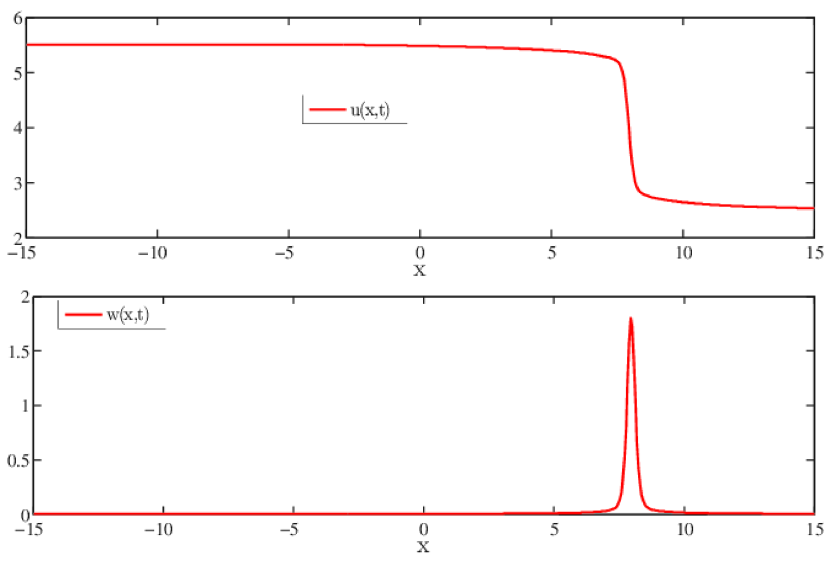
$$\partial_t h + \partial_x(hu) = 0$$
$$\partial_t(hu) + \partial_x(hu^2 + \frac{g}{2}h^2) = 0$$
$$W = -\frac{g}{2}h^2\partial_x u$$

Can be proved that the previous systems yield the correct shock propagation and therefore as $\varepsilon \to 0$: if $\partial_x u = 0$ then $W = 0$, if $\partial_x u \to -\infty$ then $W \to \infty$ (shock wave) and if $\partial_x u \to \infty$ then $W \to -\infty$ (rarefaction wave). Consequently, $W(x,t)$ tends to δ -function located at the points of the discontinuities.

Numerical Experiments

Using developing shock initial conditions in both cases, we proof numerically that for a sufficiently small epsilon ($\varepsilon = 0.01$), $W(x,t)$ behaves as expected.

1-D Inviscid Burgers' equation:
$u(x,0) = \tan^{-1}(-x) + 4 \quad \forall x$



Figur 1. $u(x,t)$ and $w(x,t)$ at $t = 2$

1-D Shallow Water equations:
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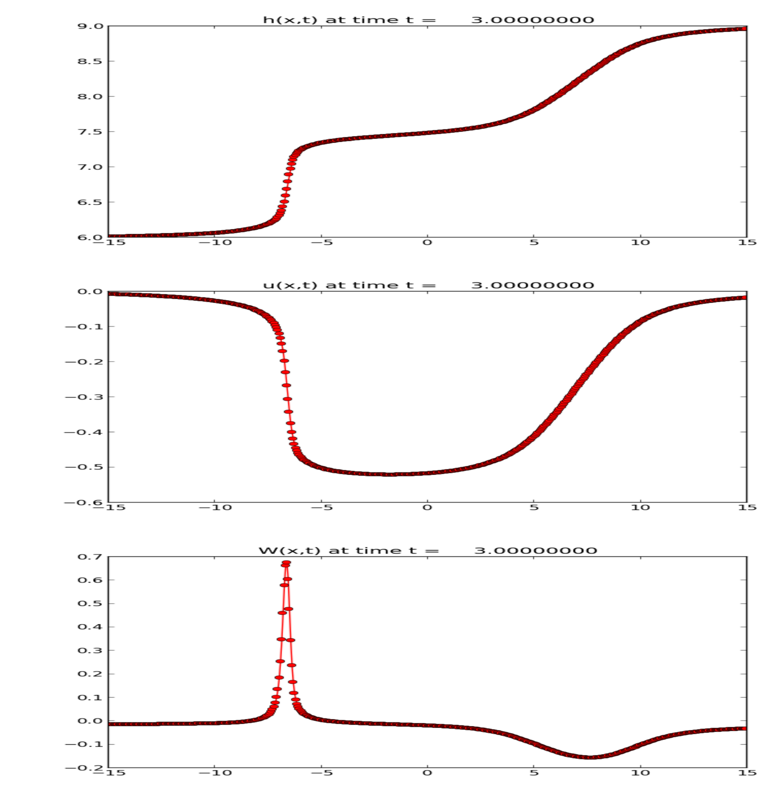

Figur 2. $h(x,t)$, $u(x,t)$ and $w(x,t)$ at $t = 3$

Present and Future Work

- The current stage of our research is focused on the applicability of the monitoring function as a refinement parameter in the construction of novel grid-adaptive simulation tools. Again, we use the previous base cases in the development of numerical experiments and compare its performance with traditional grid-adaptive simulations.
- The next step consist in the derivation of novel parametrizations for subgrid closures. We will use spectral analysis in order to compare in a coherent way the original flow equations and its corresponding moment system, together with the corresponding subgrid closures.

Selected References

[1] G. Diana and J. Struckmeier: *Kinetic-Induced Moment Systems for the Saint-Venant Equations*, TASK QUARTERLY, 2013, Vol. 17 No. 1-2, pp. 6390.
[2] G. Diana, H. Struchtrup and J. Struckmeier: *A Kinetic-Induced Moment System for Burgers Equation*, 2014 (in preparation).
[3] B. Perthame: *Kinetic Formulation of Conservation Laws*, OXFORD University Press, NY, 2002.
[4] B. Perthame and C. Simeoni: *"A Kinetic Scheme for the Saint-Venant System with a source term"*, CALCOLO, 2001, No. 38, pp. 201-231.

[5] H. Struchtrup: *Macroscopic Transport Equations for Rarefied Gas Flows*, Springer-Verlag, Berlin, 2005.

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