**Project Aims and Challenges**

This project focuses on the derivation of Kinetic-Induced Moment Systems (KIMS) based on the relationship between kinetic theory and non-linear hyperbolic conservation laws and their potential applications in the description of small-scale geophysical flows. Using simple base cases as the 1-D Burgers’ equation and the 1-D shallow water equations, our aim on the one hand is to use the resulting PDEs as a monitoring function to detect particular flow structure (like shocks and rarefaction waves) into the construction of adaptive numerical methods, and on the other hand as a basis to derive novel parametrizations for subgrid closures.

**Main Definitions**

- **Boltzmann-like equation with a BGK collision term:** describes the statistical distribution of the density of particles \( f(t, x, \xi) \), where \( f_0(Q, \xi) \) is considered its equilibrium function and \( \xi \) the microscopic velocity. 
  \[
  \partial_t f(t, x, \xi) + \xi \partial_x f(t, x, \xi) = \frac{1}{\epsilon} f_0(Q, \xi) - f(t, x, \xi)
  \]
  with \( 0 < \epsilon \ll 1 \), the mean free path.

- **Moments (\( W_k \)):** are weighted averages of \( f(t, x, \xi) \).
  \[
  W_k = \int \xi^k f(t, x, \xi) d\xi
  \]
  with \( k \in \mathbb{Z} \), the order of the moment.

The equilibrium moments \( W_0 \) are also weighted averages but of \( f_0(Q, \xi) \).
\[
W_0 = \int \xi^0 f_0(Q, \xi) d\xi
\]
Consider a 1-D conservation law system
\[
\partial_t Q + \partial_x F(Q) = 0
\]
\( Q(n) \) the vector of unknowns with \( n \) elements. For \( 0 < k \leq n-1 \) holds that \( f(t, x, \xi) = f_0(Q, \xi) \) and \( W_{k-1} = W_{k-1} \mid Q = Q(k), \quad 0 < k \leq n-1 \) and the remaining moments \( W_k \) for \( k \geq n \), will yield the new unknowns.

**What is a KIMS?**

It is an infinite moment system based on an artificial Boltzmann-like transport equation using the connection between kinetic theory and conservation laws together with an asymptotic expansion of the corresponding moments.

**How to derive it?**

1. **Multiplication on both sides of the Boltzmann-like transport equation by the weights (1, \( \xi, \ldots, \xi^{n-1}, \xi^n \)) and subsequent integration over the microscopic velocity \( \xi \) yield an infinite PDE moment system.

2. **Express the infinite system in terms of the \( p \)-order non-equilibrium moments, starting at \( p = 1 \):**

   \[
   W_k^{(p)} = W_k^{(p-1)} - W_{k-1}^{(p-1)} \mid Q, \quad p \geq 1
   \]
   for \( 0 \geq k \geq n + p - 2 \) holds that \( W_k^{(p)} = 0 \).

3. **Use asymptotic expansion in terms of the small parameter \( \epsilon \) to show that at each order a scale-induced closure is possible, resulting in a closed moment system.**

   \[
   W_k^{(p)} = \epsilon^p W_{k,p}^{(0)} + \epsilon^{p+1} W_{k,p+1}^{(1)} + \ldots, \quad k \geq n + p - 1
   \]

**Base Cases**

1. **1-D Inviscid Burgers’ equation:**

   \[
   \partial u + \partial_x \left( \frac{1}{2} u^2 \right) = 0
   \]

   evolution of the horizontal velocity \( u(t, x) \) at a time \( t \geq 0 \) and at a point \( x \in \mathbb{R} \).

2. **1-D Shallow Water equations:**

   \[
   \partial_h + \partial_x (hu) = 0
   \]

   \[
   \partial_t h + \partial_x (hu^2 + \frac{1}{2} g h^2) = 0
   \]

   evolution of the height of water \( h(t, x) \) and its horizontal velocity \( u(t, x) \), with \( t \geq 0 \), \( x \in \mathbb{R} \) and \( g \) the gravitational acceleration.

The respective KIMS at third order (\( p = 3 \)) reads,

**Monitoring Functions (\( \epsilon \to 0 \))**

In the limit \( \epsilon \to 0 \), the previous third-order systems read

1. **1-D Inviscid Burgers’ equation:**

   \[
   \partial u + u \partial_x u = 0
   \]

   \[
   W = -\frac{1}{3} u
   \]

2. **1-D Shallow Water equations:**

   \[
   \partial_h + \partial_x (hu) = 0
   \]

   \[
   \partial_t h + \partial_x (hu^2 + \frac{1}{2} g h^2) = 0
   \]

   \[
   W = -\frac{2}{3} h/\partial_x u
   \]

Can be proved that the previous systems yield the correct shock propagation and therefore as \( \epsilon \to 0 \). If \( \partial_x u \not= 0 \) then \( W \not= 0 \) and if \( \partial_x u \to -\infty \) then \( W \to -\infty \) (shock wave) and if \( \partial_x u \to \infty \) then \( W \to -\infty \) (rarefaction wave). Consequently, \( W(t, x) \) tends to \( \delta \)-function located at the points of the discontinuities.

**Numerical Experiments**

Using developing shock initial conditions in both cases, we proof numerically that for a sufficiently small epsilon (\( \epsilon = 0.01 \)), \( W(t, x) \) behaves as expected.

**Present and Future Work**

- **The current stage of our research is focused on the applicability of the monitoring function as a refinement parameter in the construction of novel grid-adaptive simulation tools.** Again, we use the previous base cases in the development of numerical experiments and compare its performance with traditional grid-adaptive simulations.

- **The next step consist in the derivation of novel parametrizations for subgrid closures.** We will use spectral analysis in order to compare in a coherent way the original flow equations and its corresponding moment system, together with the corresponding subgrid closures.

**Selected References**


**Affiliations**

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