# Models in Geophysical Fluid Dynamics in Nambu Form

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## *Overview*

- Nambu‘s (1973) extension of Hamiltonian mechanics
- Hydrodynamics in Nambu form (Névir and Blender 1993)
- Conservative codes (Salmon 2005)
- Review on applications of GFD
- Perspectives and Summary

# Nambu and Hamiltonian mechanics

Hamiltonian dynamics

Basic ingredient: Liouville's theorem

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Nambu mechanics (Nambu 1973)
two stream-functions *C* and *H*

$$\dot{x}_i = \sum_{j,k} \epsilon_{ijk} \frac{\partial C}{x_j} \frac{\partial H}{x_k}$$

Yōichirō Nambu
Nobel prize 2008

## Low dimensional Nambu systems

Nambu (1973)

Euler equations rigid top

Névir and Blender (1994)

Lorenz-1963 conservative parts

Chatterjee (1996)

Isotropic oscillator, Kepler problem, Vortices

Phase space geometry and chaotic attractors

Axenides and Florates (2010), Roupas (2012)

# The Lorenz Equations in Nambu form

Conservative dynamics

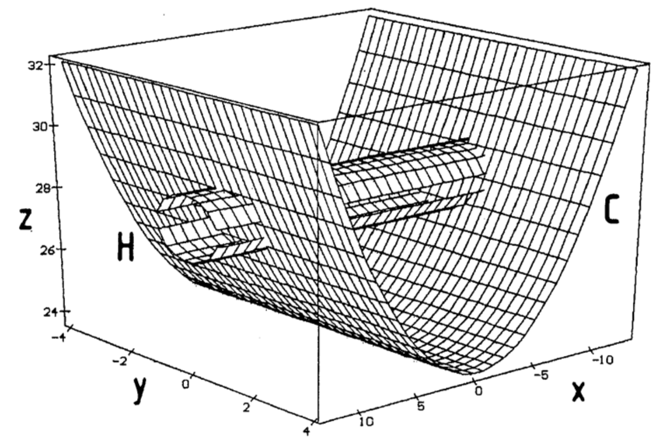
$$\dot{x} = \sigma y$$

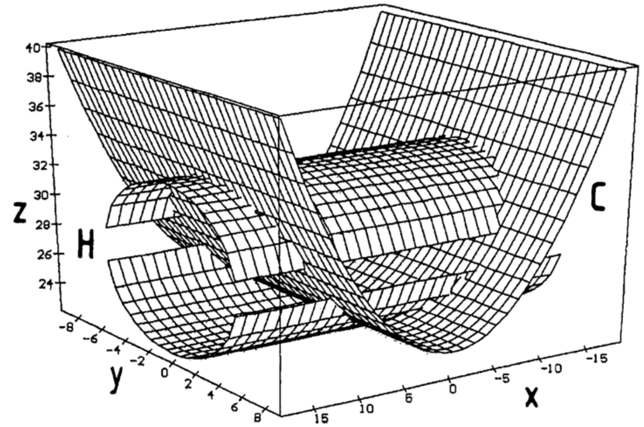
$$\dot{y} = rx - xz$$

$$\dot{z} = xy$$

Conservation laws

$$C = \frac{1}{2}x^2 - \sigma z \qquad H = \frac{1}{2}\left(y^2 + z^2\right) - rz$$




Nambu

$$\dot{\mathbf{X}} = \nabla C \times \nabla H$$

Névir and Blender (1994)

# Minos Axenides and Emmanuel Floratos (2010)
# Strange attractors in dissipative Nambu mechanics

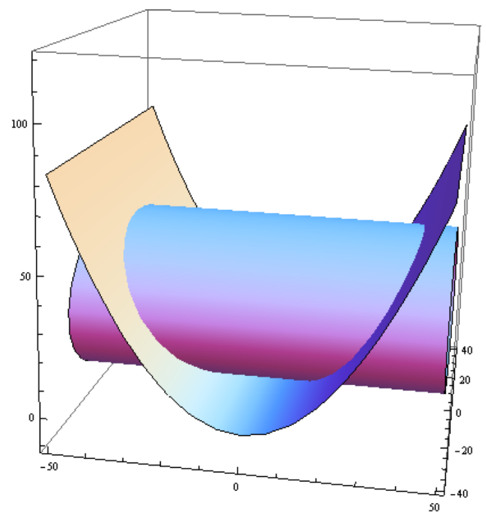



**Figure 1.** Intersecting surfaces for Lorenz Attractor.

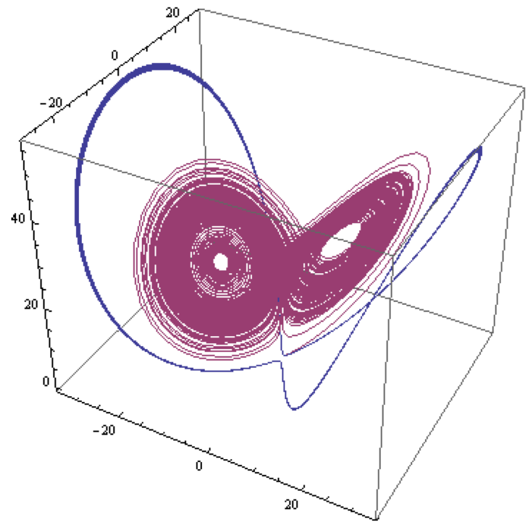



**Figure 2.** Nondissipative Orbit and Lorenz Attractor.

# Point vortices

Kirchhoff 1876

$$\Gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = -\frac{\partial H}{\partial y_i}, \quad \Gamma_i \frac{\mathrm{d}y_i}{\mathrm{d}t} = \frac{\partial H}{\partial x_i} \qquad H = -\frac{1}{4\pi} \sum_{\substack{i \neq j \\ i,j=1}}^{N} \Gamma_i \Gamma_j \ln(r_{ij})$$

Nambu form for three point vortices (Müller and Névir 2014)

Conservation

$$M = -\Gamma L_z - \frac{1}{2}(P_x^2 + P_y^2) = \frac{1}{4} \sum_{\substack{i \neq j \\ i,j=1}}^{N} \Gamma_i \Gamma_j r_{ij}^2$$

$$\frac{\mathrm{d}r_{ij}}{\mathrm{d}t} = \frac{\sigma}{2\Gamma_1\Gamma_2\Gamma_3\rho} \left( \frac{\partial M}{\partial r_{jk}} \frac{\partial H}{\partial r_{ki}} - \frac{\partial M}{\partial r_{ki}} \frac{\partial H}{\partial r_{jk}} \right)$$

$$\rho \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t'} = \nabla M \times \nabla H \qquad \rho := \rho(r_{ij}, r_{jk}, r_{ki}) = \frac{r_{ij} r_{jk} r_{ki}}{4A_{ijk}}$$

# 2D Vorticity equation: Nambu Representation

$$\frac{\partial \zeta}{\partial t} = \mathcal{J}\left(\zeta, \psi\right) \qquad \mathcal{J}(a,b) = \partial_x a\, \partial_y b - \partial_y a\, \partial_x b$$

Nambu: Enstrophy and energy as Hamiltonians

$$\boxed{\frac{\partial \zeta}{\partial t} = -\mathcal{J}\left(\frac{\delta \mathcal{E}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta \zeta}\right)} \qquad \frac{\delta}{\delta \zeta}\mathcal{H} = -\psi$$

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial t} &= -\int \frac{\delta \mathcal{F}}{\partial \zeta}\mathcal{J}\left(\frac{\delta \mathcal{E}}{\delta \zeta}, \frac{\delta \mathcal{H}}{\delta \zeta}\right) \mathrm{d}A \\ &= \{\mathcal{F}, \mathcal{E}, \mathcal{H}\}\end{aligned}$$

cyclic $\quad \{\mathcal{F}, \mathcal{E}, \mathcal{H}\} = \{\mathcal{E}, \mathcal{H}, \mathcal{F}\} = \{\mathcal{H}, \mathcal{F}, \mathcal{E}\}$

(Névir and Blender, 1993)

## 3D Euler equations: Nambu Representation

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} - \boldsymbol{u} \cdot \nabla \boldsymbol{\xi}$$

Helicity and energy as Hamiltonians

$$\boxed{\frac{\partial \boldsymbol{\xi}}{\partial t} = K\left(\frac{\delta h}{\delta \boldsymbol{\xi}}, \frac{\delta H}{\delta \boldsymbol{\xi}}\right)}$$

$$K(A, B) = -\nabla \times [(\nabla \times A) \times (\nabla \times B)]$$

$$\frac{\partial \mathcal{F}}{\partial t} = -\int \left(\nabla \times \frac{\delta \mathcal{F}}{\delta \boldsymbol{\xi}}\right) \times \left(\nabla \times \frac{\delta h}{\delta \boldsymbol{\xi}}\right) \cdot \left(\nabla \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{\xi}}\right) \mathrm{d}V = \{F, h, H\}$$

(Névir and Blender, 1993)

# Hamiltonian Systems

Physical system is Hamiltonian if a Poisson bracket exists

$$\frac{\partial \mathcal{F}}{\partial t} = \{\mathcal{F}, \mathcal{H}\}_P$$

Casimir functions of the Poisson bracket (degenerate, noncanonical)

$$\{\mathcal{C}, f\}_P = 0$$ for all *f*

$$\{\mathcal{C}, H\}_P = 0$$

# Relationship between Noncanonical Hamiltonian Theory and Nambu Mechanics

Poisson and Nambu brackets

$$\{\mathcal{F}, \mathcal{H}\}_P = \{\mathcal{F}, \mathcal{E}, \mathcal{H}\}$$

Contraction/Evaluation

$$\delta\mathcal{E}/\delta\zeta = \zeta$$

- Noncanonical Hamiltonian theory is embedded in a Nambu hierarchy
- Casimirs are a second ‘Hamiltonian’
- Nambu bracket is nondegenerate (no Casimir)

# Conservative numerical codes using Nambu brackets

Salmon (2005): 2D Euler equations in Nambu form (Z enstrophy)

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H, Z\}$$

Nambu-Bracket (anti-symmetric due to the anti-symmetry of *J*)

$$\{F, H, Z\} \equiv \iint \mathrm{d}\boldsymbol{x}\, J(F_\zeta, H_\zeta) Z_\zeta$$

Rewrite bracket

$$\begin{aligned}\{F, H, Z\} &= \frac{1}{3}\iint \mathrm{d}\boldsymbol{x}[J(F_\zeta, H_\zeta)Z_\zeta + J(H_\zeta, Z_\zeta)F_\zeta + J(Z_\zeta, F_\zeta)H_\zeta] \\ &\equiv \frac{1}{3}\iint \mathrm{d}\boldsymbol{x}[J(F_\zeta, H_\zeta)Z_\zeta + \mathrm{cyc}\,(F, H, Z)],\end{aligned}$$

Numerical Conservation of *H* and *Z* (Arakawa, 1966)

- Jacobian *J*( , ) — *anti-symmetric*
- Discrete approximation of the integrals — *arbitrary*
- Energy and enstrophy — *arbitrary accuracy*
- Time stepping — *less relevant*

## Advantages of conservative algorithms

- Improve nonlinear interaction terms

- Improve energy flow across spatial scales, cascades, and avoid spurious accumulation of energy

- Impact on *nonequilibrium* flows

## Geophysical Fluid Dynamics Models

- Quasigeostrophy (Névir and Sommer, 2009)
- Rayleigh-Bénard convection (Bihlo 2008, Salazar and Kurgansky 2010)
- Shallow water equations (Salmon 2005, 2007, Névir and Sommer 2009)
- Primitive equations (Salmon 2005, Nevir 2005, 2009, Herzog and Gassmann 2008)
- Baroclinic atmosphere (Nevir and Sommer 2009)

# Quasigeostrophic Model (Névir and Sommer, 2009)

QG Potential Vorticity

$$Q = \zeta_g + \frac{f_0}{\sigma_0}\frac{\partial^2 \Phi}{\partial p^2} + f \qquad \zeta_g = 1/f_0 \nabla_h^2 \Phi$$

Conservation

$$\frac{\partial Q}{\partial t} + \frac{1}{f_0} J(\Phi, Q) = 0$$

Energy and Potential enstrophy conserved

$$\mathcal{H} = \frac{1}{2}\int \left[\left(\frac{\nabla_h \Phi}{f_0}\right)^2 + \left(\frac{1}{N}\frac{\partial \Phi}{\partial z}\right)^2\right] \mathrm{d}V \qquad \mathcal{E} = \frac{1}{2}\int Q^2 \mathrm{d}V$$

Nambu

$$\boxed{\frac{\partial Q}{\partial t} = -\mathcal{J}\left(\frac{\delta \mathcal{E}}{\delta Q}, \frac{\delta \mathcal{H}}{\delta Q}\right)}$$

Bracket

$$\frac{\partial \mathcal{F}}{\partial t} = -\int \frac{\delta \mathcal{F}}{\delta Q} J\left(\frac{\delta \mathcal{E}}{\delta Q}, \frac{\delta \mathcal{H}}{\delta Q}\right) \mathrm{d}V = \{\mathcal{F}, \mathcal{E}, \mathcal{H}\}$$

# Surface Quasi-Geostrophy and Generalizations

Blumen (1978), Held et al. (1995), Constantin et al. (1994)

## Fractional Poisson equation for active scalar and stream-function

$$\hat{q}(k) \propto -|k|^{\alpha}\hat{\psi}(k)$$

α = 1: SQG, α = 2; Euler

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0$$

## Conservation laws

1. Arbitrary function G

$$\Gamma = -\int_{R^2} G(q) d^2x$$

$$E = \frac{1}{2}\int_{R^2} q^2 d^2x$$

Kinetic Energy (SQG)

2. Total Energy (Euler)

$$H = -\frac{1}{2}\int q\psi d^2x$$

# Nambu representation for SQG and Gen-Euler

Blender and Badin (2014)

$$\frac{\partial q}{\partial t} = -J\left(\frac{\delta E}{\delta q}, \frac{\delta H}{\delta q}\right)$$

Functional derivatives $$\frac{\delta H}{\delta q} = -\psi, \quad \frac{\delta E}{\delta q} = q$$

Nambu bracket for functions F(q)

$$\frac{\partial}{\partial t} F(q) = \{F, E, H\}$$

$$\{F, E, H\} = -\int \frac{\delta F}{\delta q} J\left(\frac{\delta E}{\delta q}, \frac{\delta H}{\delta q}\right)$$

## 2D Boussinesq Approximation

Bihlo (2008), Salazar and Kurgansky (2010)

Vorticity $\omega = \nabla^2 \psi$

buoyancy

$$\frac{\partial \omega}{\partial t} = -J(\omega, \psi) - J(b, y) \qquad \frac{\partial b}{\partial t} = -J(b, \psi)$$

Conserved: Energy and helicity analogue

$$\mathcal{H} = \int \mathrm{d}^2 x \left\{ \frac{(\nabla \psi)^2}{2} - by \right\} \qquad \mathcal{G} = \int \mathrm{d}^2 x b \nabla^2 \psi$$

$$\frac{\mathrm{d}\mathcal{F}\{\omega, b\}}{\mathrm{d}t} = [\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\omega,\omega,b}$$

$$[\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\omega,\omega,b} = -\int \mathrm{d}^2 x \left\{ J\left( \frac{\delta \mathcal{F}}{\delta \omega}, \frac{\delta \mathcal{H}}{\delta b} \right) \frac{\delta \mathcal{G}}{\delta \omega} \right\} + \mathrm{cyc}(\mathcal{F}, \mathcal{H}, \mathcal{G})$$

## 3D Boussinesq approximation

Salazar and Kurgansky (2010)
Salmon (2005)

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \varpi + \mathbf{b} - \mathbf{w}_a \times \mathbf{v}$$

$$\nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial b}{\partial t} = -\mathbf{v} \cdot \nabla b$$

absolute Vorticity

$$\mathbf{w}_a = \nabla \times \mathbf{v} + 2\Omega$$

buoyancy

$$b = -g(\rho - \rho_o)/\rho_o$$

$$\mathcal{H} = \int \mathrm{d}^3x \left\{ \frac{v^2}{2} - bz \right\}$$

Energy (conserved)

$$\mathcal{G} = \frac{1}{2} \int \mathrm{d}^3x \left\{ (\nabla \times \mathbf{v} + 4\Omega) \cdot \mathbf{v} \right\}$$

Helicity (constitutive)

$$\mathcal{L} = \int \mathrm{d}^3x b.$$

Buoyancy (constitutive)

$$\frac{\mathrm{d}\mathcal{F}\{\mathbf{v}, b\}}{\mathrm{d}t} = [\mathcal{F}, \mathcal{H}, \mathcal{G}]_{\mathbf{v},\mathbf{v},\mathbf{v}} + [\mathcal{F}, \mathcal{H}, \mathcal{L}]_{b,\mathbf{v},b}$$

## Rayleigh–Bénard convection with viscous heating (2D)

Lucarini and Fraedrich (2009)

$$\partial_{t'} \nabla'^2 \psi' + J(\psi', \nabla'^2 \psi') = \sigma \partial_{x'} \theta' + \sigma \nabla'^4 \psi'$$

$$\partial_{t'} \theta' + J(\psi', \theta') = R \partial_{x'} \psi' + \nabla'^2 \theta' + \underline{\sigma \tilde{Ec} \partial_{ij} \psi' \partial_{ij} \psi'}$$

Eckert number

$$\boxed{\dot{F} = \{F, C, H\} + \langle F, \dot{C} \rangle}$$

Nambu and metric bracket

### Properties and impacts of viscous heating

- Conservation of total energy (Hamiltonian)
- Satisfies the LiouvilleTheorem
- Modifies available potential energy, source in Lorenz energy cycle
- Alters convective processes, implications for complex models

Blender and Lucarini (2011)

# Shallow water equations

Salmon (2005, 2007)
Sommer and Névir (2009)

Vorticity and divergence

$$\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\zeta_{\mathrm{a}} \mathbf{v}) \qquad \frac{\partial \mu}{\partial t} = \mathbf{k} \cdot \nabla \times (\zeta_{\mathrm{a}} \mathbf{v}) - \Delta \Psi \qquad \mu = \nabla \cdot \mathbf{v}$$

Energy

$$\mathcal{H} = \mathcal{H}_{\mathrm{kin}} + \mathcal{H}_{\mathrm{pot}}$$

$$\mathcal{H}_{\mathrm{kin}} = \int \frac{1}{2} h \mathbf{v}^2 \, \mathrm{d}A \qquad \mathcal{H}_{\mathrm{pot}} = \int \frac{1}{2} g h^2 \, \mathrm{d}A$$

Enstrophy

$$\mathcal{E} = \int \frac{1}{2} h q^2 \, \mathrm{d}A = \frac{1}{2} \int \frac{\zeta_{\mathrm{a}}^2}{h} \, \mathrm{d}A$$

Nambu

$$\begin{aligned} \partial_t \mathcal{F}[\zeta, \mu, h] &= \{\mathcal{F}, \mathcal{H}, \mathcal{E}\} \\ &= \{\mathcal{F}, \mathcal{H}, \mathcal{E}\}_{\zeta\zeta\zeta} + \{\mathcal{F}, \mathcal{H}, \mathcal{E}\}_{\mu\mu\zeta} + \{\mathcal{F}, \mathcal{H}, \mathcal{E}\}_{\zeta\mu h} \end{aligned}$$

## Shallow water equations

## Brackets

2D vorticity

$$\{\mathcal{F},\mathcal{H},\mathcal{E}\}_{\zeta\zeta\zeta} = \int J\left(\frac{\delta\mathcal{F}}{\delta\zeta},\frac{\delta\mathcal{H}}{\delta\zeta}\right)\frac{\delta\mathcal{E}}{\delta\zeta}\,\mathrm{d}A$$

$$\{\mathcal{F},\mathcal{H},\mathcal{E}\}_{\mu\mu\zeta} = \int J\left(\frac{\delta\mathcal{F}}{\delta\mu},\frac{\delta\mathcal{H}}{\delta\mu}\right)\frac{\delta\mathcal{E}}{\delta\zeta}\,\mathrm{d}A + \mathrm{cyc}(\mathcal{F},\mathcal{H},\mathcal{E})$$

$$\begin{aligned}\{\mathcal{F},\mathcal{H},\mathcal{E}\}_{\zeta\mu h}\\ &= \int \frac{1}{\partial_x q}\left(\partial_x\frac{\delta\mathcal{F}}{\delta\zeta}\partial_x\frac{\delta\mathcal{H}}{\delta\mu} - \partial_x\frac{\delta\mathcal{F}}{\delta\mu}\partial_x\frac{\delta\mathcal{H}}{\delta\zeta}\right)\partial_x\frac{\delta\mathcal{E}}{\delta h}\,\mathrm{d}A\\ &+ \int \frac{1}{\partial_y q}\left(\partial_y\frac{\delta\mathcal{F}}{\delta\zeta}\partial_y\frac{\delta\mathcal{H}}{\delta\mu} - \partial_y\frac{\delta\mathcal{F}}{\delta\mu}\partial_y\frac{\delta\mathcal{H}}{\delta\zeta}\right)\partial_y\frac{\delta\mathcal{E}}{\delta h}\,\mathrm{d}A\\ &+\mathrm{cyc}(\mathcal{F},\mathcal{H},\mathcal{E}).\end{aligned}$$

Flow

$$h\mathbf{v} = \mathbf{k}\times\nabla\chi + \nabla\gamma \qquad \Psi = \tfrac{1}{2}\mathbf{v}^2 + gh$$

Derivatives

$$\frac{\delta\mathcal{H}}{\delta\zeta} = -\chi, \qquad \frac{\delta\mathcal{H}}{\delta\mu} = -\gamma, \qquad \frac{\delta\mathcal{H}}{\delta h} = \Psi,$$

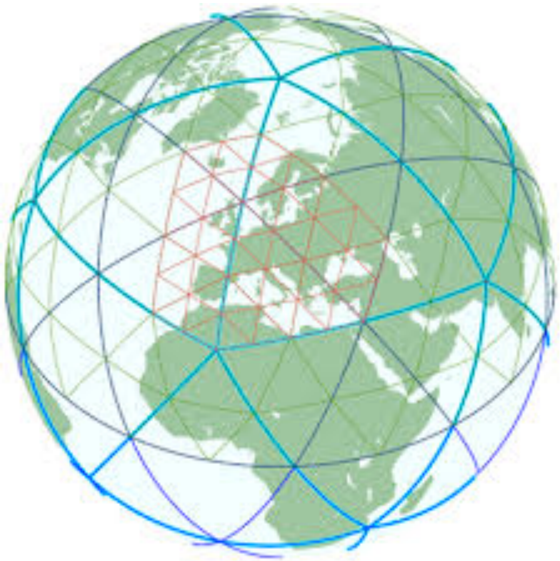
$$\frac{\delta\mathcal{E}}{\delta\zeta} = q, \qquad \frac{\delta\mathcal{E}}{\delta\mu} = 0, \qquad \frac{\delta\mathcal{E}}{\delta h} = -\frac{1}{2}q^2$$

# Global Shallow Water Model using Nambu Brackets

Sommer and Névir (2009)

ICON model (Isosahedric grid, Non-hydrostatic, German Weather Service and Max Planck Institute for Meteorology, Hamburg).

## ICON grid structure

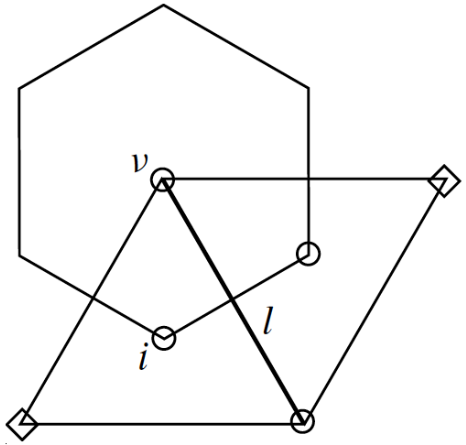


## Variables

$v$ vorticity, vertices

$i$ mass. triangle centers

$l$ wind, edges




Grid: functional derivatives, operators (div and curl), Jacobian and Nambu brackets

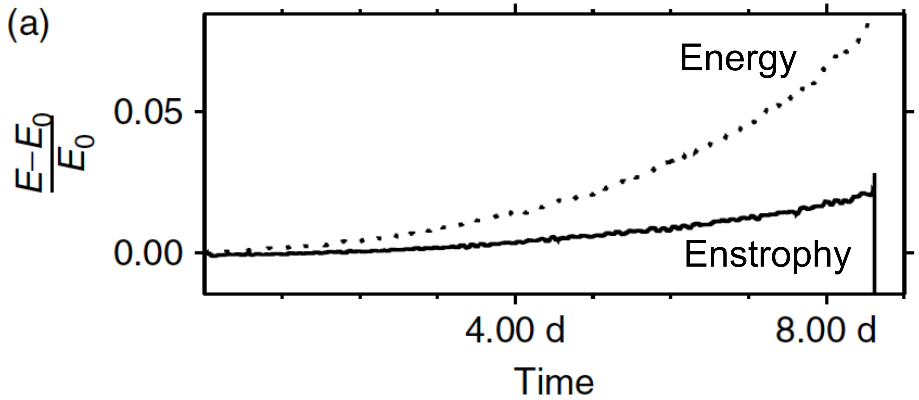
Time step: leap-frog with Robert-Asselin filter

# Computational instability

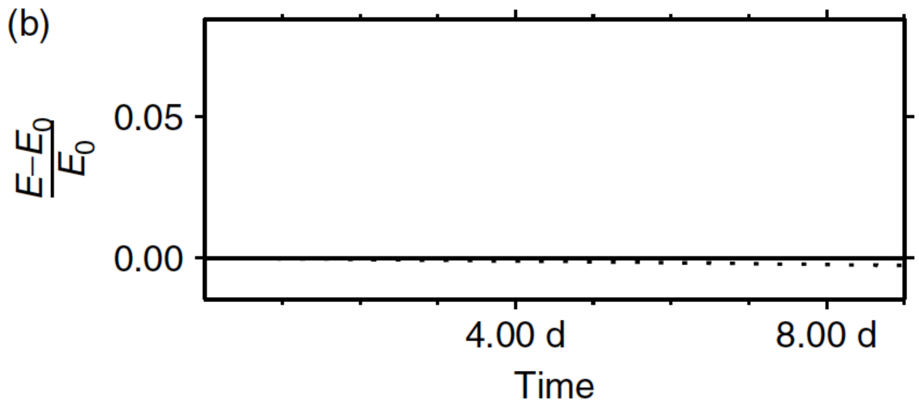
## Energy and potential enstrophy

ICOSWP

*potential enstrophy conserving*




Nambu



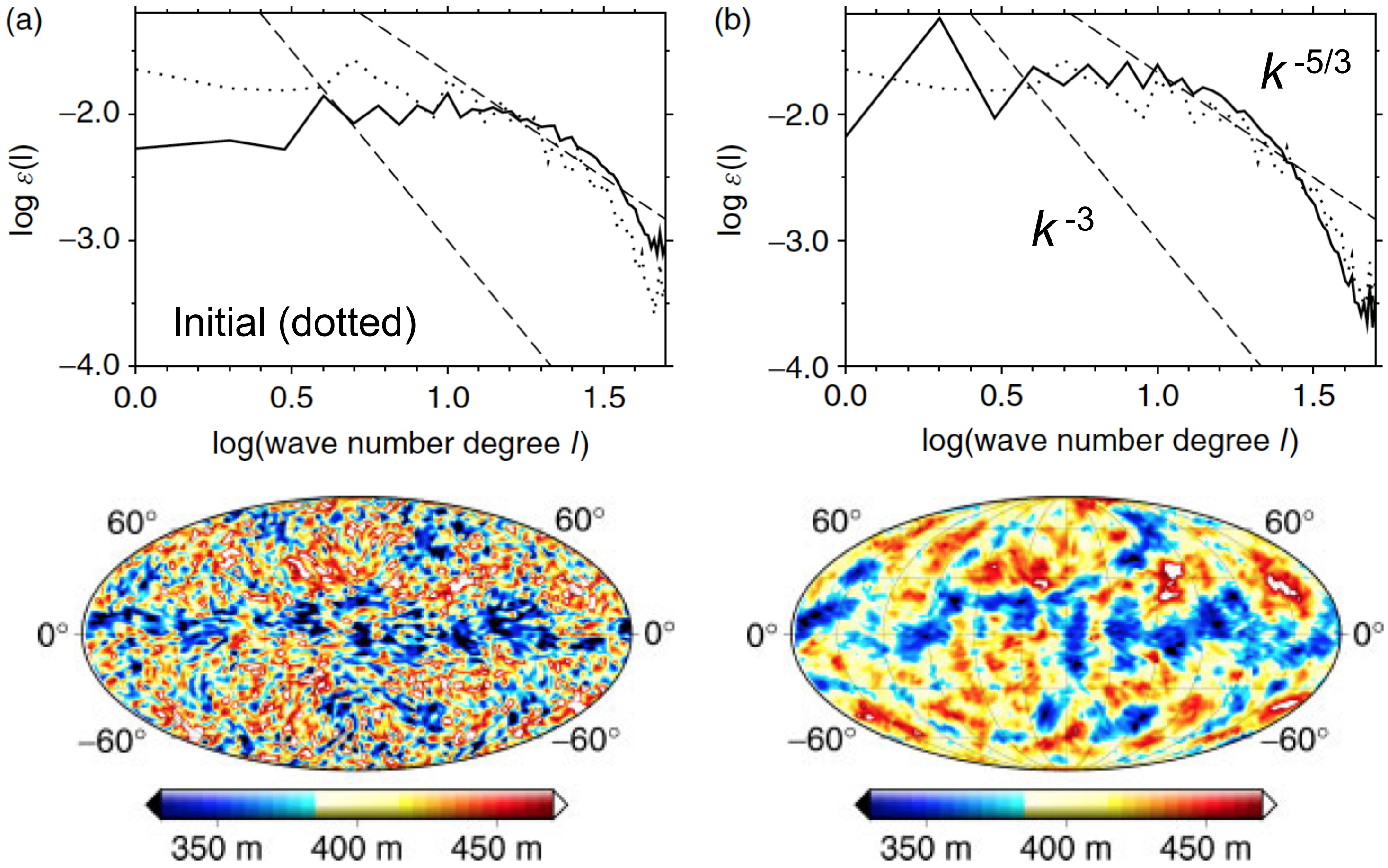

Sommer and Névir (2009)

# Energy spectra and height snapshots (decaying)

ICOSWP | Nambu

Kinetic Energy

Height




Sommer and Névir (2009)

# Baroclinic Atmosphere

Névir (1998), Névir and Sommer (2009), Herzog and Gassmann (2008)

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} - 2\boldsymbol{\omega} \times \mathbf{v} - \frac{1}{\rho} \nabla p - \nabla \phi^{(s)}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}),$$

$$\frac{\partial s}{\partial t} = -\mathbf{v} \cdot \nabla s.$$

Energy $$\mathcal{H}[\mathbf{v}, \rho, s] = \int_V d\tau \left\{ \frac{1}{2} \rho \mathbf{v}^2 + \rho e[\upsilon(\rho), s] + \rho \phi^{(s)} \right\}$$

Helicity $$h_a = \frac{1}{2} \int_V d\tau \mathbf{v}_a \cdot \boldsymbol{\xi}_a$$

Mass, entropy $$\mathcal{M} = \int_V d\tau \rho, \quad \mathcal{S} = \int_V d\tau \rho s$$

# Baroclinic Atmosphere as a Modular Nambu-system

$$\frac{\partial \mathbf{v}}{\partial t} = \{\mathbf{v}, h_a, \mathcal{H}\}_h + \{\mathbf{v}, \mathcal{M}, \mathcal{H}\}_m + \{\mathbf{v}, \mathcal{S}, \mathcal{H}\}_s$$

3D incompr

$$\frac{\partial \rho}{\partial t} = \{\rho, \mathcal{M}, \mathcal{H}\}_m$$

barotropic

$$\frac{\partial \sigma}{\partial t} = \{\sigma, \mathcal{S}, \mathcal{H}\}_s$$

baroclinic

Casimir $$\mathcal{C}_\Psi[\mathbf{u}, \rho, s] = \int_V d\tau \rho \Psi(\Pi, s) \quad \text{and} \quad \Pi = \frac{\boldsymbol{\xi}_a \cdot \nabla s}{\rho}.$$

Névir (1998), Névir and Sommer (2009)

## Gassmann and Herzog (2008)

### Towards a consistent numerical code using Nambu brackets

- Compressible non-hydrostatic equations
- Turbulence-averaged
- Dry air and water in three phases
- Poisson bracket for full-physics equation

## Gassmann (2012)

Non-hydrostatic core (ICON) with energetic consistency

$$\frac{\partial \mathcal{F}}{\partial t} = \{\mathcal{F}, \mathcal{H}\} + (\mathcal{F}, \mathbf{f}_\mathbf{r}) + (\mathcal{F}, Q)$$

$$(\mathcal{F}, \mathbf{f}_\mathbf{r}) = -\int_V \frac{\delta \mathcal{F}}{\delta \mathbf{v}} \cdot \frac{1}{\varrho} \nabla \cdot \overline{\varrho \mathbf{v}'' \mathbf{v}''} d\tau \qquad \text{turbulent friction}$$

$$(\mathcal{F}, Q) = \int_V \frac{\delta \mathcal{F}}{\delta \tilde{\theta}} \frac{\varepsilon}{c_p \pi} d\tau \qquad \text{frictional heating}$$

# Dynamic equations from conservation laws

## A perspective for modelling and parameterizations

**Standard approach**

Equations

Identify
conservation laws

**A Nambu approach**

Conservation laws
and operators

Equations

## Summary

- Application of Nambu Mechanics in Geophysical Fluid Dynamics

  Second ‘Hamiltonian’ due to particle relabeling symmetry

  Includes noncanonical Hamiltonian fluid dynamics (Casimirs)

- Flexible approach (constitutive integrals)

  Modular design and approximations

- Construction of conservative numerical algorithms

  Improves nonlinear terms, energy and vorticity cascades

*Thank you*

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# Appendix

## Nambu brackets (Takhtajan 1994)

Skew symmetry (permutations σ with parity ε)

$$\{f_1, \ldots, f_n\} = (-1)^{\varepsilon(\sigma)}\{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\}$$

Leibniz rule

$$\{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + f_2\{f_1, f_3, \ldots, f_{n+1}\}$$

Fundamental (Jacobi) Identity

$$\{\{f_1, \ldots, f_{n-1}, f_n\}, f_{n+1}, \ldots, f_{2n-1}\} + \{f_n, \{f_1, \ldots, f_{n-1}, f_{n+1}\}, f_{n+2}, \ldots, f_{2n-1}\}$$
$$+ \ldots + \{f_n, \ldots, f_{2n-2}, \{f_1, \ldots, f_{n-1}, f_{2n-1}\}\} = \{f_1, \ldots, f_{n-1}, \{f_n, \ldots, f_{2n-1}\}\},$$

**Hierarchy of Nambu brackets**

Example: Poisson bracket with Casimir *C* from Nambu bracket

$$\{F_1, F_2, C\} = \{F_1, F_2\}$$

# Jacobi Identity and contraction

Salmon (2005)

Consider Nambu bracket

$$\{F, H, Z\} \equiv \iint \mathrm{d}\boldsymbol{x}\, J(F_\zeta, H_\zeta) Z_\zeta$$

Contraction with H (evaluate)

$$\{F, Z\} = \iint \mathrm{d}\boldsymbol{x}\, \psi\, J(F_\zeta, Z_\zeta)$$

This contraction does not obey the Jacobi Identity for Poisson brackets

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0$$

Applications require only that the Nambu brackets are anti-symmetric
Consequence: Not insist on the generalized (Nambu) Jacobi Identity

## Types of Nambu Representations

Roberto Salazar and Michael Kurgansky (2010), see also Nambu (1973)

**Nambu brackets first kind NBI:**
different triplets, same conserved quantities

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_k \frac{\partial(F, H, G)}{\partial(x_k, y_k, z_k)} = \sum_k [F, H, G]_k$$

**Second kind NB II:**
also *constitutive* quantities
not necessarily constants of motion

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \sum_i \frac{\partial(F, H_i, G_i)}{\partial(x, y, z)} = \sum_i [F, H_i, G_i]$$

Advantages:

- Flexibility in the construction of a Nambu representation
- Modular decomposition and approximation

# Nonconservative forces: Metriplectic systems

Kaufman (1984), Morrison (1986), Bihlo (2008)
Metriplectic systems conserve energy but are irreversible

Brackets: symplectic and metric

$$\frac{dA}{dt} = \{A, H\} + \langle A, S \rangle$$

Entropy S: Casimir of the symplectic bracket

$$\{S, H\} = 0, \qquad \{H, H\} = 0$$

Entropy increases

$$\langle H, S \rangle = 0, \qquad \langle S, S \rangle \geq 0$$