# Perturbations of N-Vortex Equilibria


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Workshop on Fundamentals of Climate, Atmosphere and Ocean Dynamics
Hamburg, May 12-14, 2014

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# The Main Object of Study

Source: http://en.wikipedia.org/wiki/Vortex

## Definition

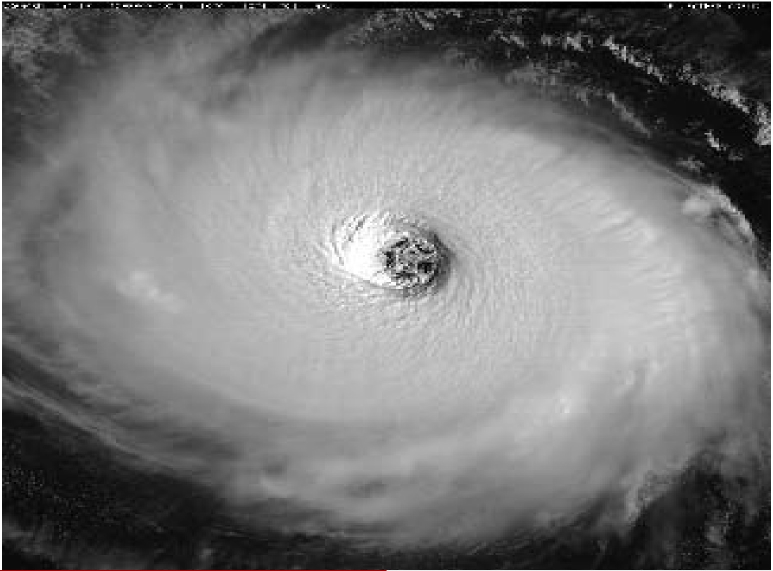
*Vorticity* describes the tendency of a fluid to spin, or swirl. Formally, it is related to the velocity field **u** by $\omega = \nabla \times \mathbf{u}$.

## Definition

A coherent *vortex* is a localized region of enhanced vorticity.

# Example of Vortex Equilibrium

Vortex relative equilibrium in a hurricane [Kossin and Schubert, 2004]



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**Consider the motion of a collection of well-separated, disjoint vortex patches in the plane governed by the two-dimensional incompressible Euler equations.**

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- As the initial diameter tends to zero, the vorticity distribution converges weakly to a collection of Dirac masses (point vortices)
- The motion of these small patches is governed by the “point vortex equations”

# The Euler Point Vortex Equations of Motion

Let $q_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 0, 1, ..., N$ denote the positions of $N$ point vortices with circulation $\Gamma_i$.

$$\dot{q}_j = \sum_{i=0, i \neq j}^{N} \Gamma_i \frac{(q_j - q_i)^{\perp}}{|q_j - q_i|^2} \tag{1}$$

where $(x, y)^{\perp} = (-y, x)$.

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**Remark**

This system is Hamiltonian with "interaction energy"

$$H(q_0, ..., q_N) = -\sum_{i<j} \Gamma_i \Gamma_j \log |q_i - q_j|,$$

and so we expect the system to have conserved quantities and exhibit symmetries.

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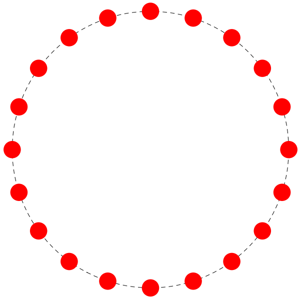
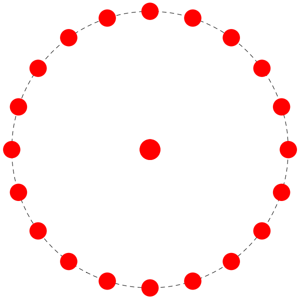
In this talk, we focus on **relative equilibria**. Some known examples of **rigidly rotating relative equilibria** are the $N$- and $1 + N$-gon configurations.





# Background: Spectral Stability for Hamiltonian Systems

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### Punchline

**An equilibrium of a Hamiltonian system can only possess a weak form of stability**.
Eigenvalues of the linearization come in real or complex pairs, or complex quartets.

# Spectral Stability of the *N*- and $1 + N$-gon Configurations

- **Thomson** 1882. The *N*-gon is **spectrally stable** for $N < 7$, **degenerate** for $N = 7$ and **unstable** for $N \geq 8$.

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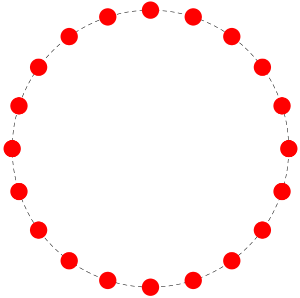
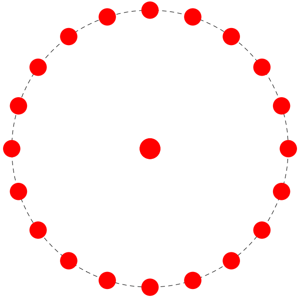
- **Cabral, Schmidt** 1999. Consider the $1 + N$-gon with central vortex having strength 1 and outer vortices having equal strength $\Gamma$. There is an interval in $\Gamma$ for which the configuration is locally **nonlinearly stable** (and the configuration is unstable outside of this interval).





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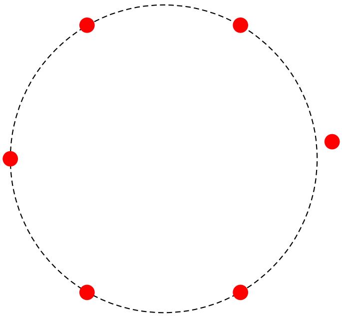

# Example ($N = 3$)

## Perturbing the Equilateral Triangle

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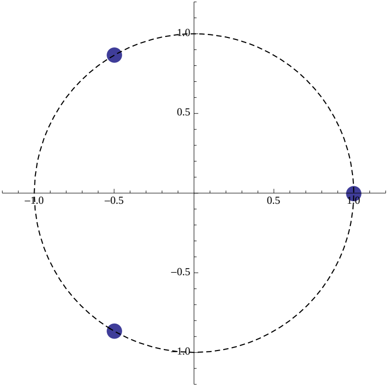


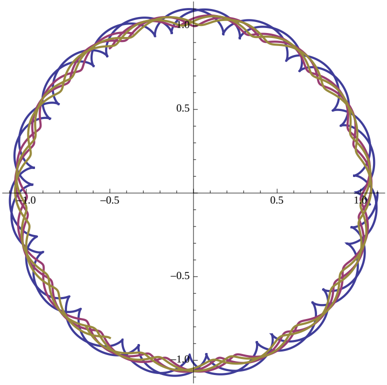

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- Now $F_i$ and $G_i$ are functions of $\tilde{r}_i$, $\tilde{\theta}_i$ and $\varepsilon$
- Use the **Implicit Function Theorem** to prove that zeroes of $F_i$ and $G_i$ persist for $|\varepsilon| > 0$ sufficiently small.

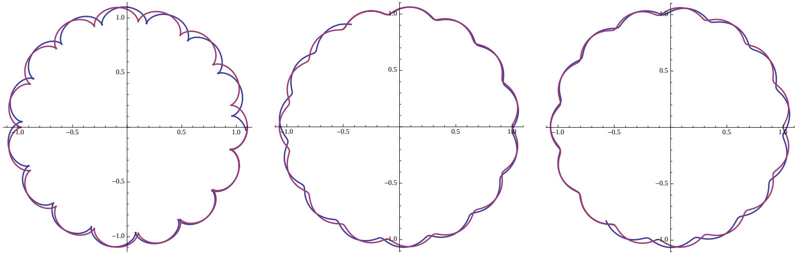
# Back to $N = 3$

For $N = 3$, we calculate the $\mathcal{O}(\varepsilon)$ terms in the expansions explicitly. We have $r_i(t) = 1 + \varepsilon\tilde{r}_i(t) + \mathcal{O}(\varepsilon^2)$, $\theta_i(t) = \frac{2(k-1)\pi}{3} + \varepsilon\tilde{\theta}_i(t) + \mathcal{O}(\varepsilon^2)$, $k = 1, 2, 3$ with

$$\tilde{r}_1(t) = \frac{1}{3}(1 + 2\cos(t)), \quad \tilde{r}_2(t) = \tilde{r}_3(t) = \frac{1}{3}(1 - \cos(t))$$

$$\tilde{\theta}_1(t) = -\frac{2}{3}(t + \sin(t)), \quad \tilde{\theta}_2(t) = \tilde{\theta}_3(t) = \frac{1}{3}(-2t + \sin(t))$$

Below we compare the actual vortex motion to the first order terms in the expansions:



# Passive Tracer Motion

Now suppose we track the motion of a **passive tracer** under the influence of ***N*-point vortices** of **equal strength** $\Gamma$ in an $N$-gon configuration with **rotation rate** $\omega$:

$$\dot{x} = \omega y - \sum_{i=1}^{N} \Gamma_i \frac{y - y_i}{(x - x_i)^2 + (y - y_i)^2}$$

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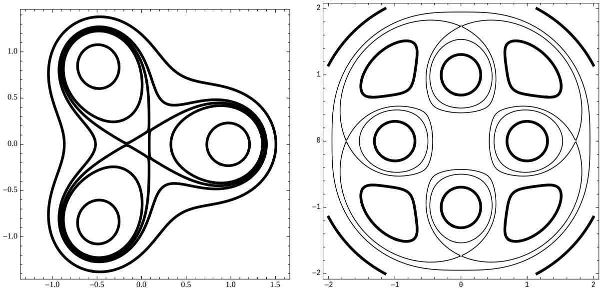
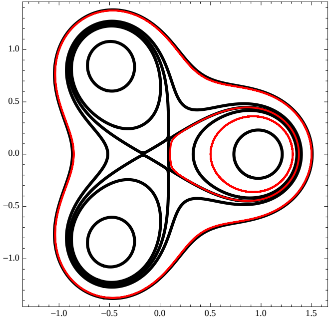


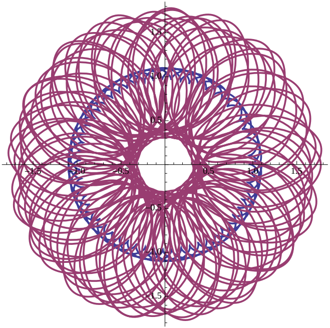
# Passive Tracer- $N = 3$

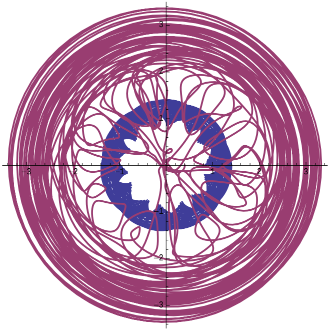
- When $\varepsilon = 0$, the tracer motion is restricted to streamlines.
- The 3-vortex problem is **completely integrable**, so there can be **no chaotic motion of vortices** upon perturbations.



# Passive Tracer- $N = 3$

- For $\varepsilon > 0$, the tracer motion becomes more complicated.
- Tracer motion in the 3-vortex problem can be chaotic (Kuznetsov and Zaslavsky 2000).



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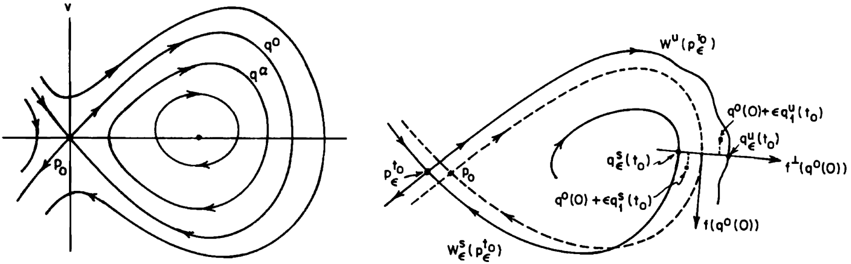
Can the tracer motion be chaotic, even if $\varepsilon$ is very small?
**Idea:** Apply Melnikov Theory- measures distance between stable and unstable manifolds




from Guckenheimer and Holmes 1983

Images

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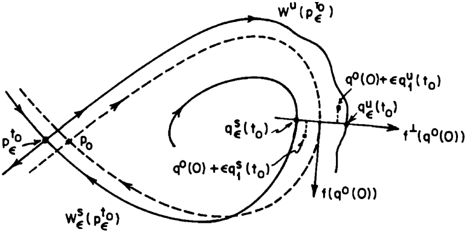


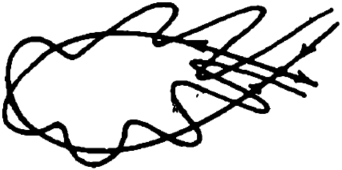

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- Punchline: The Melnikov distance can easily be calculated numerically in individual cases, but may not be tractable analytically unless we can exploit the symmetries of the $N$-gon configuration.

# Visions of the Future

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- Can we quantify transport and mixing that occurs when separatrices break or intersect transversally?

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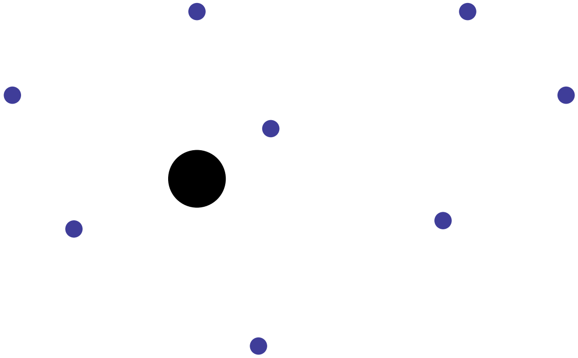
**Thank you (and stay tuned)!** If time, go on to next problem...

# The 1 + $N$-Vortex Problem

**Goal.** Study existence and stability of relative equilibria in the problem of 1 strong vortex and $N$ weak vortices



# Relative Equilibrium of the $(1 + N)$-Vortex Problem

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4. **Fixed rotation rate**: $\omega = 1$

# A Property of Limit Configurations

Under assumptions 1-4, one can prove the

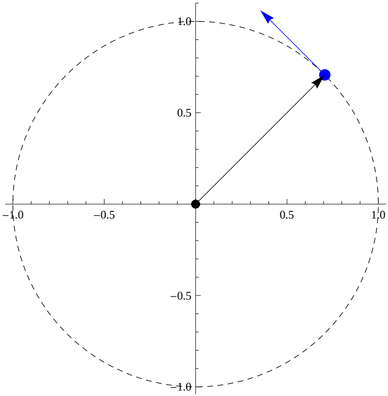
**Lemma 1.1[Barry et al., 2011].**

All relative equilibria, $\mathbf{x}_0^\varepsilon, \ldots, \mathbf{x}_N^\varepsilon$, which converge to a relative equilibrium $\mathbf{x}_0, \ldots, \mathbf{x}_N$ of the $(1+N)$-vortex problem satisfy $|\mathbf{x}_0^\varepsilon| = \mathcal{O}(\varepsilon)$, $|\mathbf{x}_j^\varepsilon|^2 - 1 = \mathcal{O}(\varepsilon)$, $j = 1, \ldots, N$.

1+3 Vortex relative equilibria

# Relative Equilibria of the Limit Problem

**Observation**: Periodic relative equilibria with center of vorticity at the origin must satisfy $\mathbf{x}_j \cdot \dot{\mathbf{x}}_j = 0$ for each $j$



# Limit Potential

Using the observation and Lemma 1, one can prove the following

**Lemma 1.2 [Barry et al., 2011].**

All periodic relative equilibria of the $(1+N)$-vortex problem, $N \geq 2$, are critical points of

$$V(\theta_1, ..., \theta_N) = -\sum_{i<j}\left(\cos(\theta_i - \theta_j) + \frac{1}{2}\log(2 - 2\cos(\theta_i - \theta_j))\right).$$

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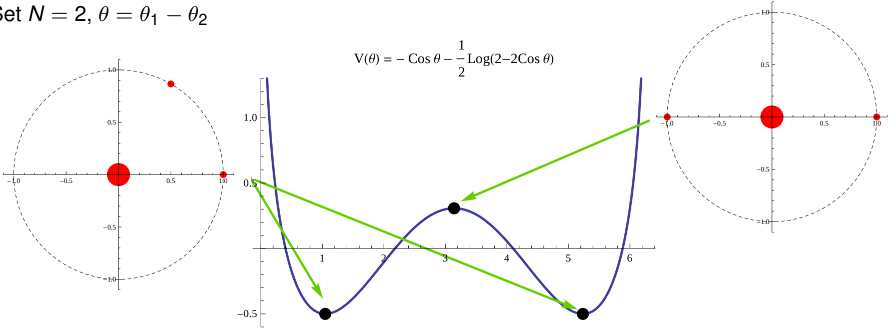
Set $N = 2$, $\theta = \theta_1 - \theta_2$




See [Newton, 2001]

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**Q: Are all critical points of $V$ limits of sequences of relative equilibria of the full problem as $\varepsilon \to 0$?**

# $\varepsilon \neq 0$: Continuation of Critical Points and Linear Stability

**Theorem 1.1 [Barry et al., 2011]**

"Nondegenerate" critical points of $V$ can be continued to relative equilibria of the full problem with $\varepsilon \neq 0$ sufficiently small.

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A family of relative equilibria which converges to a nondegenerate relative equilibrium $\theta$ of the $(1+N)$-vortex problem is

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# Sketch of Proof of Theorem 1.2

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**Key Observation:** The matrix of the linearization, M, is given by

$$M = \begin{pmatrix} -\varepsilon A + \mathcal{O}(\varepsilon^2) & \varepsilon V_{\theta\theta}(\phi) + \mathcal{O}(\varepsilon^2) \\ -2I + \mathcal{O}(\varepsilon) & \varepsilon A + \mathcal{O}(\varepsilon^2) \end{pmatrix}$$

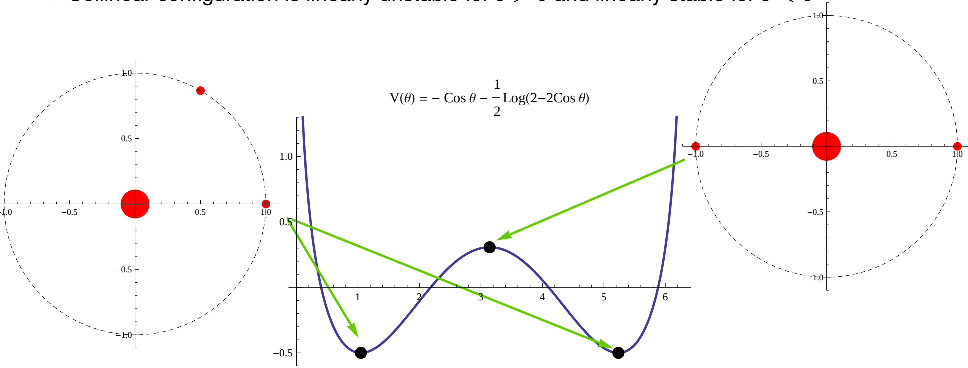
and after some simplification,

$$\det(M - \lambda I) = (1 + \mathcal{O}(\sqrt{\varepsilon}))\det(\lambda^2 I + 2\varepsilon V_{\theta\theta} + \mathcal{O}(\varepsilon^{5/4})).$$

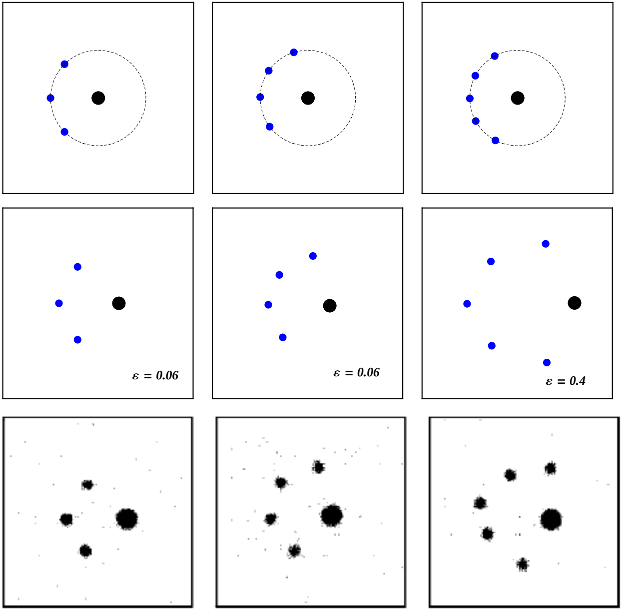
# A Low-Dimensional Example

$$N = 2$$

- Theorem 1.1: Critical points can be continued to relative equilibria for $\varepsilon \neq 0$ sufficiently small
- Theorem 1.2: Equilateral triangle is linearly stable for $\varepsilon > 0$ and linearly unstable for $\varepsilon < 0$
- Collinear configuration is linearly unstable for $\varepsilon > 0$ and linearly stable for $\varepsilon < 0$



# Numerical Continuation of Minima of $V$ for $N = 3, 4, 5$



# The Minimum Family: Predictions for Further Electron Column Experiments?

- The **minimum** of $V$ continues to a **linearly stable** family of relative equilibria when $\varepsilon > 0$.
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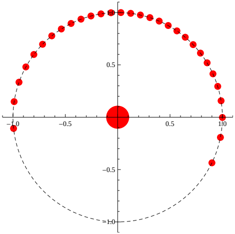
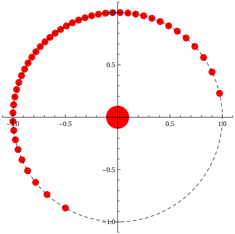
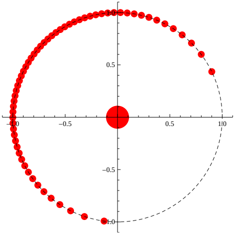



$N = 45$




$N = 60$



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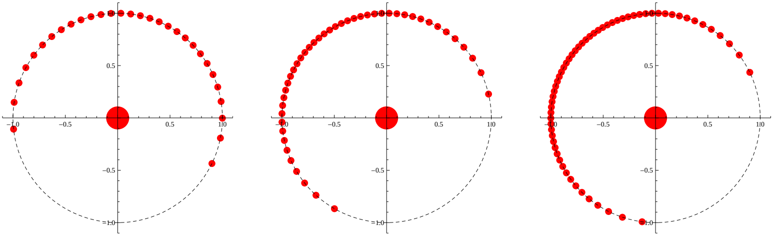
**A1: Theorem 1.3 [Barry et al., 2011]**

Under nondegeneracy assumptions, $V$ has at least three families of critical points for all $N \geq 4$, one of which is a *minimum*.

Idea of Proof:

- One family consists of $(1 + N)$-gons
- Structure of limit potential $\Rightarrow$ $V$ must have a minimum
- Hopf Index Theorem: there must be a third critical point with negative index

$N = 30$    $N = 45$    $N = 60$



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**A2: Theorem 1.4 [Barry, 2012]**

All symmetric "limiting distributions" of the minimum configuration are members of the family

$$f^{\alpha}(\theta) = \frac{1}{2\pi} + \alpha \cos\theta, \quad |\alpha| \le \frac{1}{2\pi}.$$

Idea of Proof:

- $V$ is undefined in the limit $N \to \infty$. Instead we choose a limiting functional and use standard Fourier analysis to show $f^{\alpha}$ is the unique minimizer.

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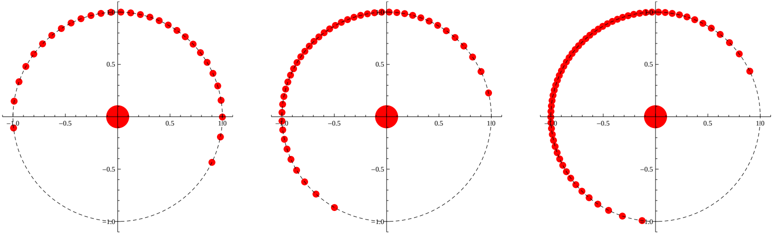

# The Hurricane Example: Instability of the $(1 + N)$-gon

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- The matrix $V_{\theta\theta}$ is *circulant* allowing for explicit computation of its eigenvalues
- For $N \geq 4$, $\lambda_1 = 0$, $\lambda_{2,3} = -\frac{1}{2}$, $\lambda_j > 0$ for $j \geq 4$
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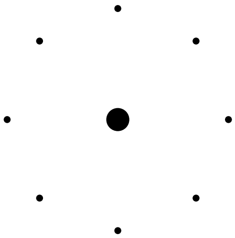

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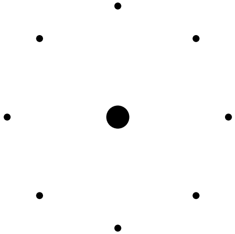
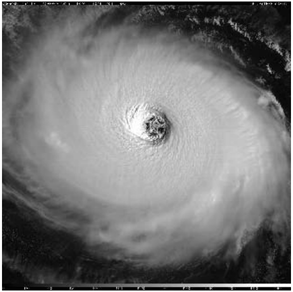
**Remark.** In contrast, the $(1 + N)$-gon is a **linearly stable** relative equilibrium of the $(1 + N)$-body problem for $N \geq 7$.



# So what about Hurricane Isabel?



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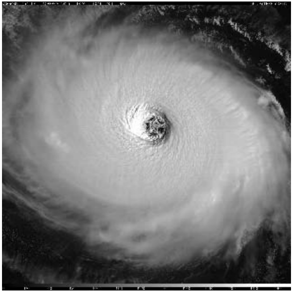

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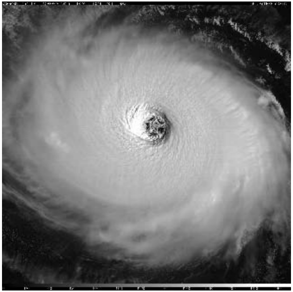

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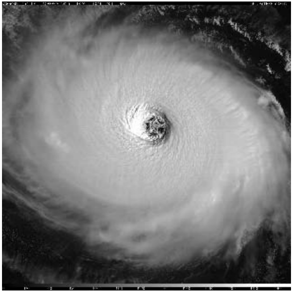

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$$\frac{N^2-8N+8}{16} < p < \frac{(N-1)^2}{4} \quad \text{for N even,}$$
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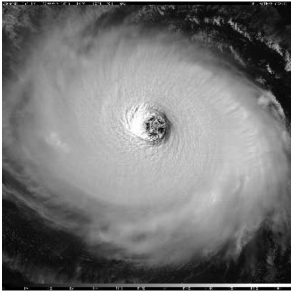

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- Using $\Gamma=\varepsilon$, $p=\frac{1}{\varepsilon}$, this yields *instability* for $-2<\varepsilon<\frac{1}{4}$ .

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