Application of WKB theory for the simulation of the weakly nonlinear dynamics of gravity waves

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Motivation: GW Impacts

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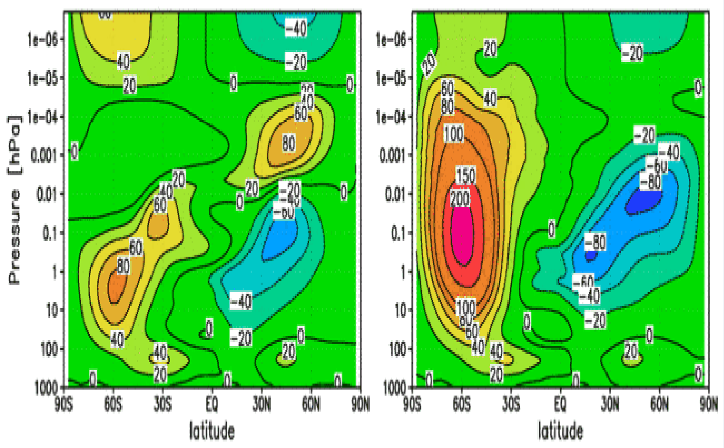
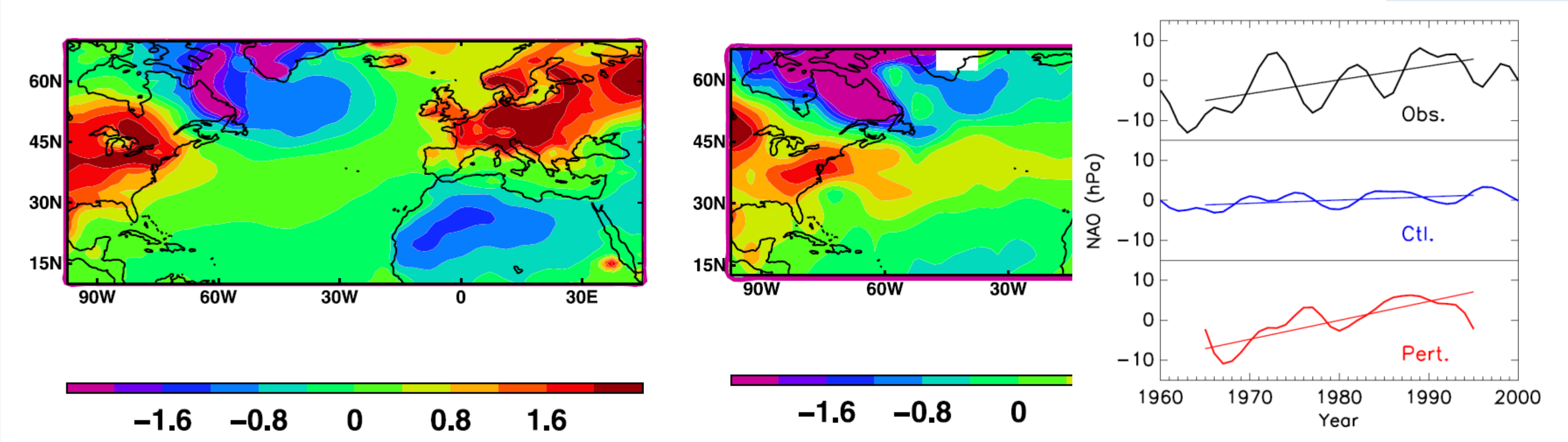
Gravity-wave effects numerous, e.g.

- **Clear-air turbulence (e.g. Koch et al 2005)**
- **Clouds & convection (e.g. Zhang et al 2001, 2003)**
- **Middle-atmosphere waves (QBO, solar tides)**
- **residual circulation**
  - GW impact in stratosphere (e.g. Palmer et al 1986)
  - GW control in mesosphere (e.g. Lindzen 1981)
- **Indirectly: Impact middle atmosphere on troposphere (downward control)**

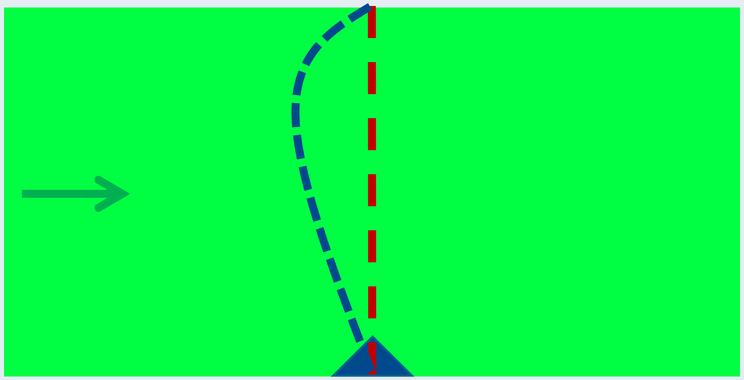


Schmidt et al (2006)



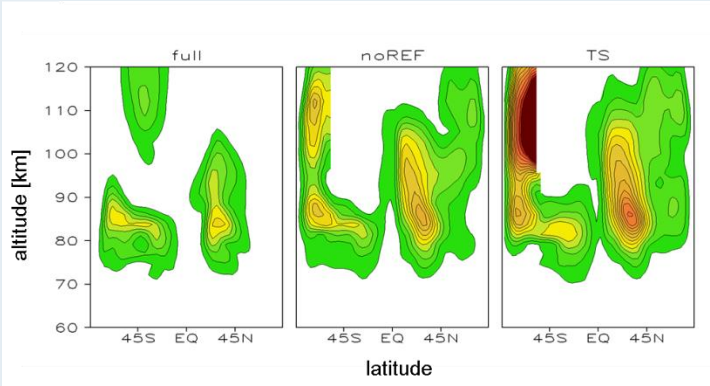
Scaife et al (2005)

Motivation: Parameterization of GW Processes



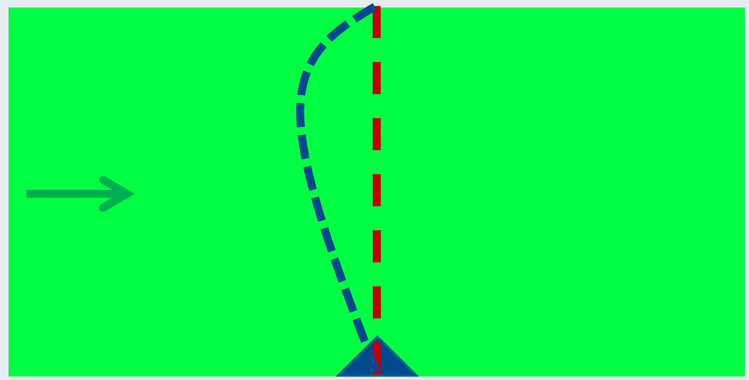
GW propagation

- **Described using WKB theory (Grimshaw 1975, …)**
- **Simplifications for efficiency:**
  - **Single-column**
  - **Steady state**
  - Transience considered important (intermittency, Alexander et al 2010)
  - Horizontal propagation has an effect (Dunkerton1984, …, Kawatani et al 2010)
- **Synoptic-scale balanced background assumed**
  - But NWP models resolve some GWs
  - Theory to be revisited



Effects on GW-ST IA (Senf & Achatz 2011)

Motivation: Parameterization of GW Processes



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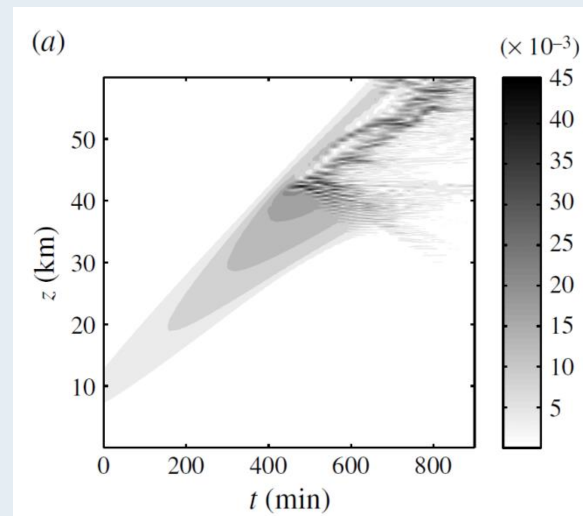
Nonlinear dissipation

- **accelerated by wave-mean flow interaction (Dosser & Sutherland 2011)**



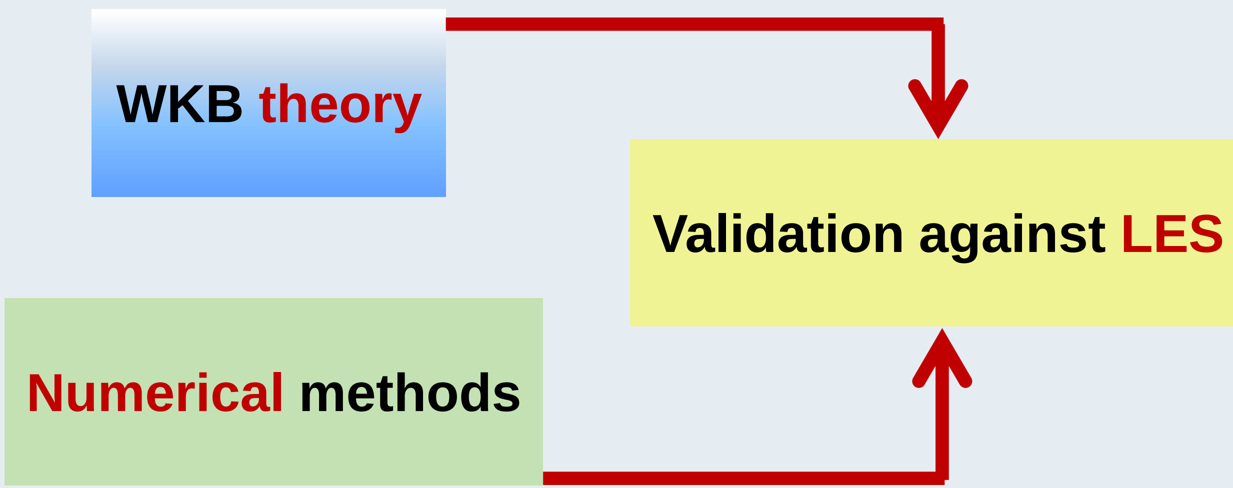
Rieper, Achatz & Klein (2013)

Objectives & Strategy

Goal:

- **A prognostic WKB model for propagation and dissipation of subgrid-scale GWs**
- **to be implemented into NWP and climate models.**

Strategy:

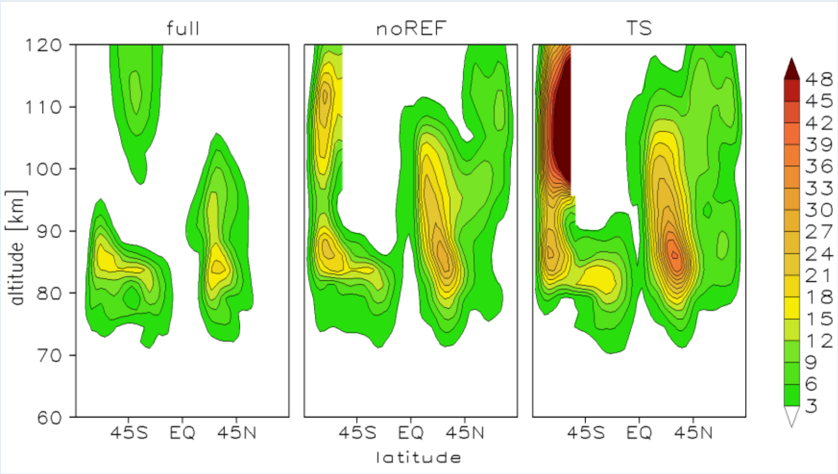


Preliminary Work

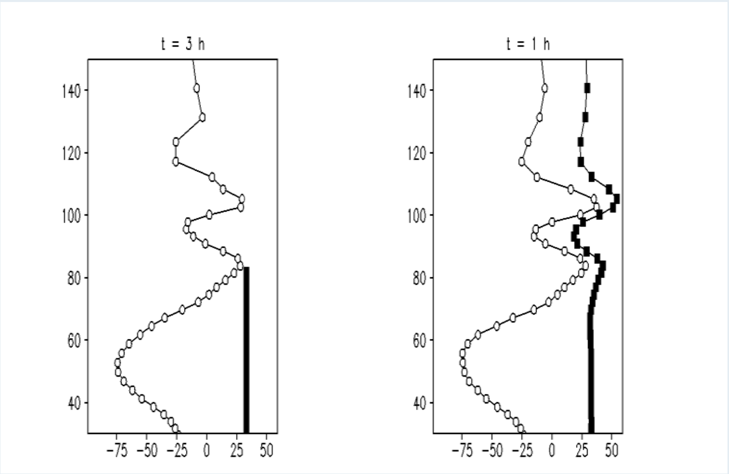
LES code pseudo-incompressible equations (Rieper, Hickel & Achatz 2013)

4D ray tracing (Senf & Achatz 2011)

- **applied to interaction GWs with solar tides**
- **single column and steady state lead to overestimation of GW impact**
- **refraction by horizontal gradients leads to considerable latitudinal displacements**



z (km)



true

no transience

u and c (m/s)

no hor. refraction

suppression of critical layers

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

- **Scaling of 2D Euler so that waves are close to convective instability**
- **Scale separation parameter $\epsilon = L/H_\theta$**

$$\hat{\mathbf{v}} = \tilde{\mathbf{v}}^{(0)}$$

$$\hat{\theta} = \hat{\theta}^{(0)} + \varepsilon \tilde{\theta}^{(1)}$$

$$\hat{\pi} = \hat{\pi}^{(0)} + \varepsilon^2 \tilde{\pi}^{(2)}$$

$$\tilde{\mathbf{v}}^{(0)} = \hat{\mathbf{V}}^{(0)} + \varepsilon \hat{\mathbf{V}}^{(1)} + o(\varepsilon) \quad \text{(e.g.)}$$

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$$\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\underbrace{\varepsilon\hat{t}}_{\tau}, \underbrace{\varepsilon\hat{x}}_{\chi}, \underbrace{\varepsilon\hat{z}}_{\zeta}) + \Re\left\{\hat{\mathbf{V}}_1^{(0)}(\tau,\chi,\zeta)\exp\left[\frac{i}{\varepsilon}\varphi(\tau,\chi,\zeta)\right]\right\}$$

Mean flow with only large-scale dependence

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$$\mathbf{k} = \nabla_{(\chi,\zeta)}\varphi$$

$$\omega = -\frac{\partial\varphi}{\partial\tau}$$

Wavepacket with amplitude, wavenumber and frequency with large-scale dependence

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$$\hat{\mathbf{V}}^{(1)} = \hat{\mathbf{V}}_0^{(1)}(\tau,\chi,\zeta) + \Re\sum_{\alpha=1}^{\infty}\left\{\hat{\mathbf{V}}_\alpha^{(1)}(\tau,\chi,\zeta)\exp\left[\frac{i}{\varepsilon}\alpha\varphi(\tau,\chi,\zeta)\right]\right\}$$

Next-order mean flow

Large-Amplitude WKB

(Achatz, Klein & Senf 2010)

- **Scaling of 2D Euler so that waves are close to convective instability**
- **Scale separation parameter $\epsilon = L/H_\theta$**

$$\hat{\mathbf{v}} = \tilde{\mathbf{v}}^{(0)}$$

$$\hat{\theta} = \hat{\theta}^{(0)} + \varepsilon\tilde{\theta}^{(1)}$$

$$\hat{\pi} = \hat{\pi}^{(0)} + \varepsilon^2\tilde{\pi}^{(2)}$$

$$\tilde{\mathbf{v}}^{(0)} = \hat{\mathbf{V}}^{(0)} + \varepsilon\hat{\mathbf{V}}^{(1)} + o(\varepsilon) \quad \text{(e.g.)}$$

$$\hat{\mathbf{V}}^{(0)} = \hat{\mathbf{V}}_0^{(0)}(\underbrace{\varepsilon\hat{t}}_{\tau}, \underbrace{\varepsilon\hat{x}}_{\chi}, \underbrace{\varepsilon\hat{z}}_{\zeta}) + \Re\left\{\hat{\mathbf{V}}_1^{(0)}(\tau,\chi,\zeta)\exp\left[\frac{i}{\varepsilon}\varphi(\tau,\chi,\zeta)\right]\right\}$$

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Harmonics of the wavepacket due to nonlinear interactions

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

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Leading (zeroeth) order:

$$i\mathbf{k}\cdot\hat{\mathbf{V}}_1^{(0)} = 0$$

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M\left(\hat{\omega},\mathbf{k}\right)} \begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = 0$$

$$\hat{\omega} = \omega - k\hat{U}_0^{(0)} \qquad \text{intrinsic frequency}$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

Leading (zeroeth) order: dispersion relation and structure as from Boussinesq

$$i\mathbf{k}\cdot\hat{\mathbf{V}}_1^{(0)} = 0$$

$$\underbrace{\begin{pmatrix} -i\hat{\omega} & 0 & 0 & ik \\ 0 & -i\hat{\omega} & -N & im \\ 0 & N & -i\hat{\omega} & 0 \\ ik & im & 0 & 0 \end{pmatrix}}_{M(\hat{\omega},\mathbf{k})} \begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = 0$$

$$\hat{\omega} = \omega - k\hat{U}_0^{(0)} \qquad \text{intrinsic frequency}$$

$$\det(M) = 0 \Rightarrow$$

$$\hat{\omega}^2 = N^2 \frac{k^2}{k^2+m^2}$$

$$\begin{pmatrix} \hat{U}_1^{(0)} \\ \hat{W}_1^{(0)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(1)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(2)} \end{pmatrix} = \text{Nullvector of } M$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order:

$$
M(\hat{\omega},\mathbf{k})\begin{pmatrix} \hat{U}_1^{(1)} \\ \hat{W}_1^{(1)} \\ \frac{1}{N}\frac{\hat{\Theta}_1^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(3)} \end{pmatrix} = \begin{pmatrix} \cdots \\ \cdots \\ \cdots \\ -\frac{\partial \hat{U}_1^{(0)}}{\partial \chi} - \frac{\partial \hat{W}_1^{(0)}}{\partial \zeta} - \frac{1-\kappa}{\kappa}\frac{\hat{W}_1^{(0)}}{\hat{\pi}^{(0)}}\frac{\partial \hat{\pi}^{(0)}}{\partial \zeta} \end{pmatrix}
$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order: Solvability condition leads to wave-action conservation

$$M\left(\hat{\omega},\mathbf{k}\right)\begin{pmatrix}\hat{U}_1^{(1)}\\ \hat{W}_1^{(1)}\\ \frac{1}{N}\frac{\hat{\Theta}_1^{(2)}}{\hat{\theta}^{(0)}}\\ \hat{\theta}^{(0)}\hat{\Pi}_1^{(3)}\end{pmatrix}=\begin{pmatrix}\cdots\\ \cdots\\ \cdots\\ -\frac{\partial\hat{U}_1^{(0)}}{\partial\chi}-\frac{\partial\hat{W}_1^{(0)}}{\partial\zeta}-\frac{1-\kappa}{\kappa}\frac{\hat{W}_1^{(0)}}{\hat{\pi}^{(0)}}\frac{\partial\hat{\pi}^{(0)}}{\partial\zeta}\end{pmatrix}$$

$$\frac{\partial}{\partial\tau}\left(\frac{E'}{\hat{\omega}}\right)+\nabla_{(\chi,\zeta)}\cdot\left(\mathbf{c}_g\frac{E'}{\hat{\omega}}\right)=0$$

$$E'=\frac{\hat{\rho}^{(0)}}{2}\left(\frac{\left|\hat{\mathbf{V}}_1^{(0)}\right|^2}{2}+\frac{1}{2N^2}\left|\frac{\hat{\Theta}_1^{(0)}}{\hat{\theta}^{(0)}}\right|^2\right)\quad \text{wave energy}$$

$$\mathbf{c}_g=\left(\hat{U}_0^{(0)}+\frac{\partial\hat{\omega}}{\partial k},\frac{\partial\hat{\omega}}{\partial m}\right)\quad \text{group velocity}$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order:

$$\hat{\mathbf{V}}^{(1)} = \ldots + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{\mathbf{V}}_{\alpha}^{(1)}(\tau,\chi,\zeta) \exp\left[\frac{i}{\varepsilon} \alpha \varphi(\tau,\chi,\zeta) \right] \right\}$$

$$\alpha = 2:$$

$$M\left(2\hat{\omega}, 2\mathbf{k}\right) \begin{pmatrix} \hat{U}_2^{(1)} \\ \hat{W}_2^{(1)} \\ \frac{1}{N} \frac{\hat{\Theta}_2^{(2)}}{\hat{\theta}^{(0)}} \\ \hat{\theta}^{(0)} \hat{\Pi}_2^{(3)} \end{pmatrix} = \begin{pmatrix} \ldots \\ \ldots \\ \ldots \\ \ldots \end{pmatrix} \Rightarrow$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

1st order: 2nd harmonics are slaved

$$\hat{\mathbf{V}}^{(1)} = \ldots + \Re \sum_{\alpha=1}^{\infty} \left\{ \hat{\mathbf{V}}_{\alpha}^{(1)}(\tau,\chi,\zeta) \exp\left[\frac{i}{\varepsilon} \alpha \varphi(\tau,\chi,\zeta) \right] \right\}$$

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Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

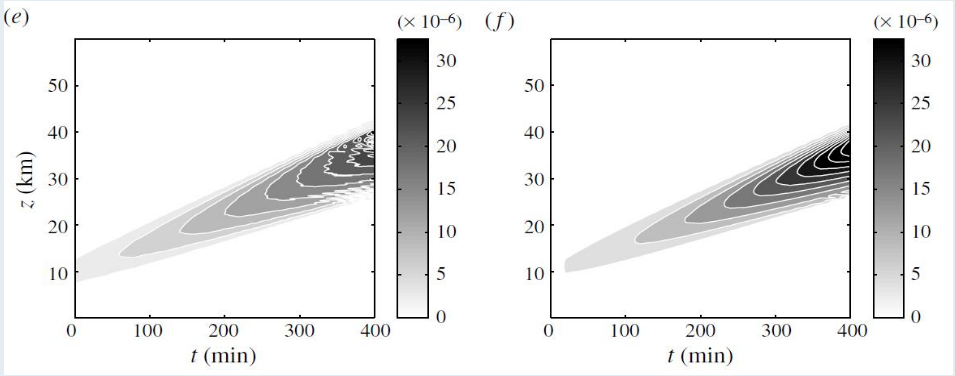
1st order: mean-flow acceleration by GW momentum-flux divergence

$$\frac{\partial \hat{U}_0^{(0)}}{\partial \tau} + \ldots = -\nabla \cdot \hat{\mathbf{F}}_{GW}^{U}$$

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

Validation (Rieper, Achatz & Klein 2013)

2nd harmonic



LES **WKB**

Large-Amplitude WKB
(Achatz, Klein & Senf 2010)

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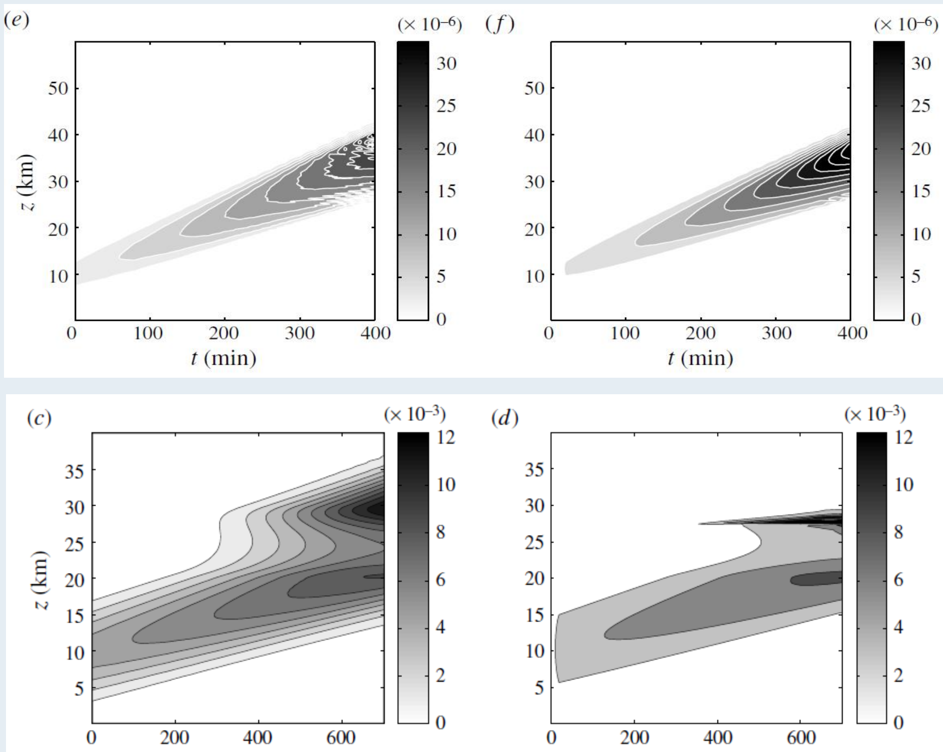


LES

WKB

Problems as soon as rays tend to cross (caustics) and GW affects the mean flow

Caustics in WKB

Summary WKB (here Boussinesq, large-scale spatial dependence 1D)

- e.g. buoyancy

$$\hat{B} = \Re\left\{b(\zeta,\tau)\, e^{i\,[kx + \phi(\zeta,\tau)/\epsilon]}\right\}$$

- unique local wavenumber and frequency

$$m(\zeta,\tau) = \frac{\partial\phi}{\partial\zeta} \qquad \omega = -\frac{\partial\phi}{\partial\tau} = kU(\zeta,\tau) \pm N(\zeta,\tau)\sqrt{\frac{k^2}{k^2+m^2}} = \Omega(\zeta,\tau,k,m)$$

- all other wave fields u, w from b, m and ω by polarization relations
- along rays

$$\frac{D_g m}{D\tau} = \frac{\partial m}{\partial\tau} + c_g\frac{\partial m}{\partial\zeta} = -\frac{\partial\Omega}{\partial\zeta} \qquad \frac{D_g\zeta}{D\tau} = c_g = \frac{\partial\Omega}{\partial m}$$

- amplitude from wave action $A = E'/\hat{\omega}$

$$\frac{D_g A}{Dt} = \frac{\partial A}{\partial t} + c_g\frac{\partial A}{\partial\zeta} = -\frac{\partial c_g}{\partial\zeta}A + D, \qquad c_g = \frac{\partial\omega}{\partial m}$$

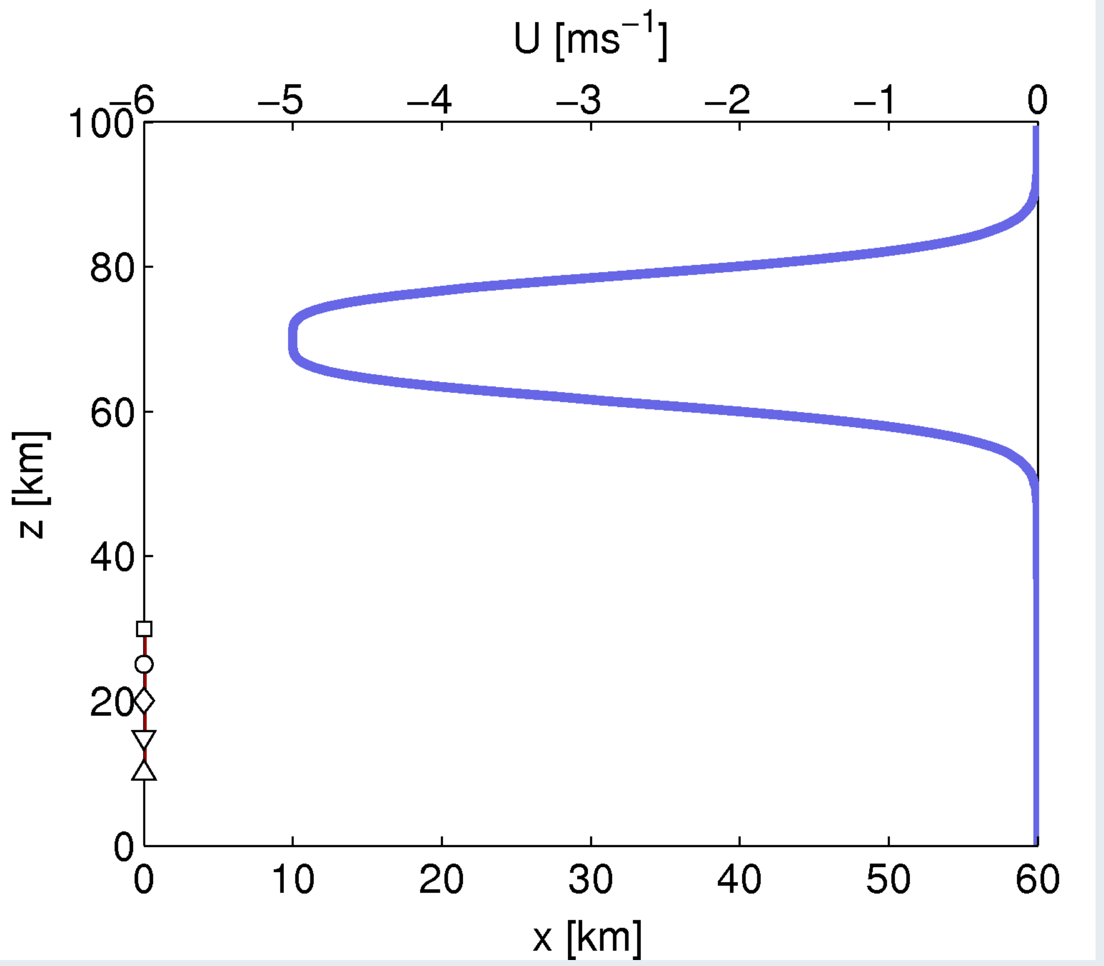
- mean wind

$$\frac{\partial U}{\partial\tau} = -\frac{\partial}{\partial\zeta}\left(\overline{uw}\right) \qquad \overline{uw} = A\, f(k,m,N)$$

Caustics in WKB

Nonuniqueness of wave number arises easily:

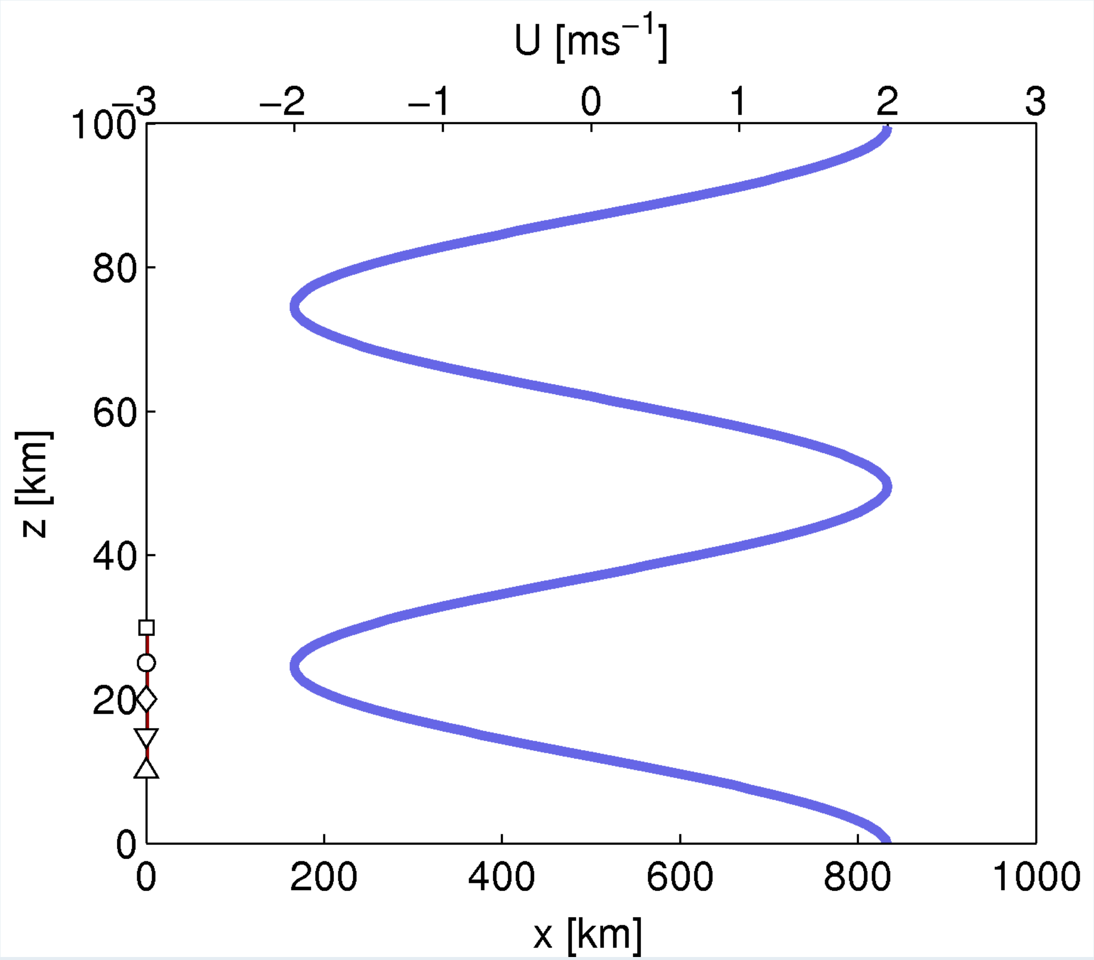
e.g. reflection at a jet



Caustics in WKB

Nonuniqueness of wave number arises easily:

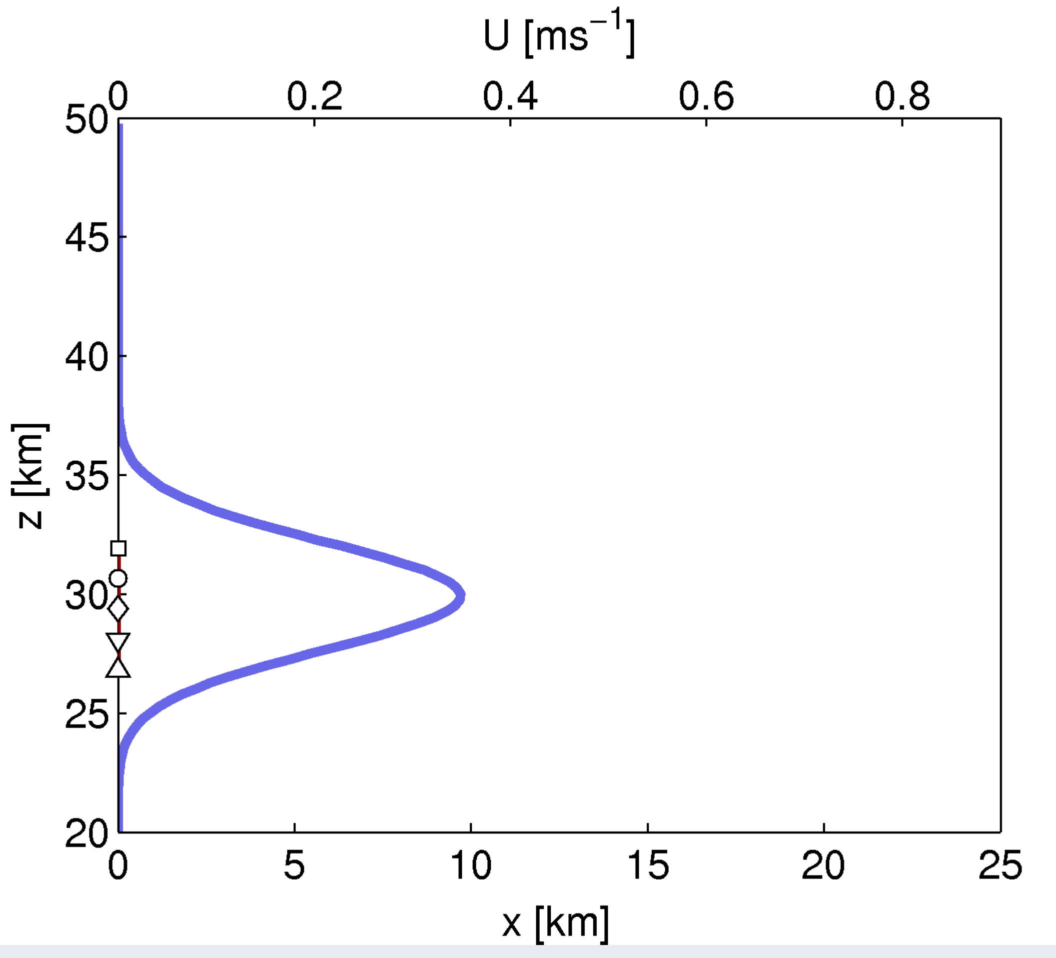
e.g. overtaking rays



Caustics in WKB

Nonuniqueness of wave number arises easily:

e.g. by
wave-induced mean flow



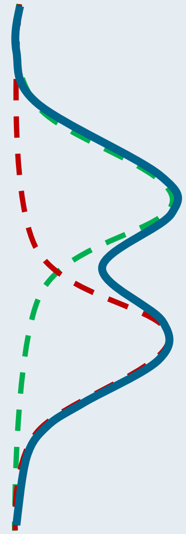
Phase-Space Wave-Action Density
(Muraschko et al 2014)

- **linear limit:**
  wave field can be decomposed into fields with singlevalued wavenumbers

$$\hat{B} = \Re\left\{ b_1(\zeta,\tau)\, e^{i\,[kx + \phi_1(\zeta,\tau)/\epsilon]} + b_2(\zeta,\tau)\, e^{i\,[kx + \phi_2(\zeta,\tau)/\epsilon]} \right\}$$

$$\frac{\partial \phi_1}{\partial \zeta} = m_1 \qquad \frac{\partial \phi_2}{\partial \zeta} = m_2$$

$$\frac{D_{g\alpha} A_\alpha}{D\tau} = \frac{\partial A_\alpha}{\partial \tau} + c_{g\alpha} \frac{\partial A_\alpha}{\partial \zeta} = -\frac{\partial c_{g\alpha}}{\partial \zeta} A_\alpha + D_\alpha \qquad (\alpha = 1, 2)$$



Case dependent surgery: very complex

Phase-Space Wave-Action Density
(Muraschko et al 2014)

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- **linear limit:**
  wave field can be decomposed into fields with singlevalued wavenumbers
- **Switch to phase space does this automatically**
  (Dewar 1970, Dubrulle & Nazarenko 1997, Bühler & McIntyre 1999, Hertzog et al 2000)

phase-space wave-action density:

$$\mathcal{N}(m,\zeta,\tau) = \int d\alpha\, A_\alpha(\zeta,\tau)\ \delta[m - m_\alpha(\zeta,\tau)]$$

conservation equation (for $D = 0$)

$$\frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta}\left(c_g\, \mathcal{N}\right) + \frac{\partial}{\partial m}\left(\dot{m}\, \mathcal{N}\right) = 0$$

mean flow by

$$\frac{\partial U}{\partial \tau} = -\frac{\partial}{\partial \zeta}\left(\overline{uw}\right) \qquad \overline{uw} = \int dm\, \mathcal{N}\, f(k,m,N)$$

generalization to 3D trivial

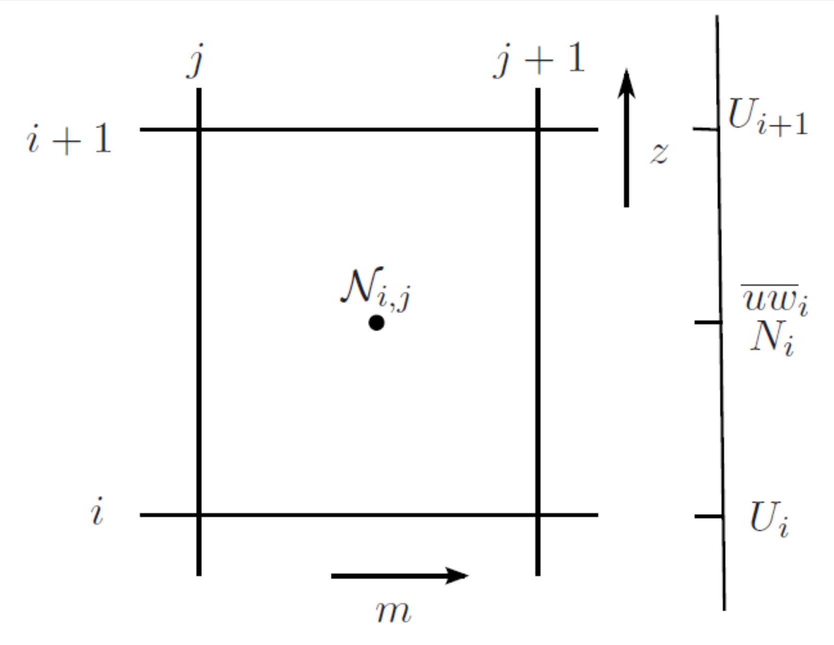
Phase-Space Wave-Action Density
(Muraschko et al 2014)

1st numerical method: Eulerian model

- **solve conservation equation**

$$\frac{\partial \mathcal{N}}{\partial \tau} + \frac{\partial}{\partial \zeta}\left(c_g\, \mathcal{N}\right) + \frac{\partial}{\partial m}\left(\dot{m}\, \mathcal{N}\right) = 0$$

on grid in phase space using finite volume scheme (MUSCL)



- **Momentum equation**

$$\frac{\partial U}{\partial \tau} = -\frac{\partial}{\partial \zeta}\left(\overline{uw}\right) \qquad \overline{uw} = \int dm\, \mathcal{N}\, f(k, m, N)$$

by finite difference

Would be too expensive in 6D!

Phase-Space Wave-Action Density
(Muraschko et al 2014)

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2nd numerical method: Lagrangian model (ray tracer)

- **Phase-space velocity is non-divergent**

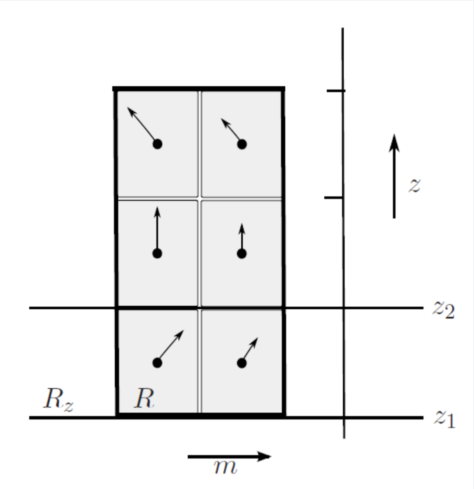
$$\frac{\partial c_g}{\partial \zeta} + \frac{\partial \dot{m}}{\partial m} = \frac{\partial}{\partial \zeta}\frac{\partial \Omega}{\partial m} + \frac{\partial}{\partial m}\left(-\frac{\partial \Omega}{\partial \zeta}\right) = 0$$

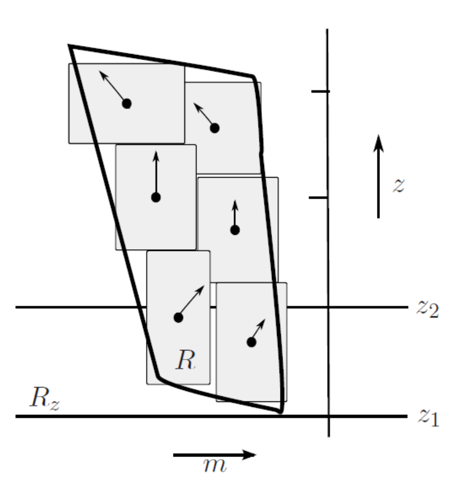
- **Hence flow is volume preserving**
- **Hence phase-space wave-action density conserved on rays**

$$\frac{D_g \mathcal{N}}{D\tau} = \frac{\partial \mathcal{N}}{\partial \tau} + c_g \frac{\partial \mathcal{N}}{\partial \zeta} + \dot{m}\frac{\partial \mathcal{N}}{\partial m} = 0$$

- **Region of nonzero $\mathcal{N}$ approximated by rectangles**
- **Rectangles move with central ray**
- **Rectangles change height ($\Delta\zeta$) and width (Δm) in area-preserving manner**

Very efficient !





Phase-Space Wave-Action Density
(Muraschko et al 2014)

Simple test case: Gaussian wave packet (a_0 = amplitude wrt static instability)

$$b'(x, z, t = 0) = A_b(z)\ \cos(kx + m_0 z)$$

$$u'(x, z, t = 0) = A_b(z) \frac{m_0}{k} \frac{\widehat{\omega}_0}{N_0^2}\ \sin(kx + m_0 z)$$

$$w'(x, z, t = 0) = -A_b(z) \frac{\widehat{\omega}_0}{N_0^2}\ \sin(kx + m_0 z)$$

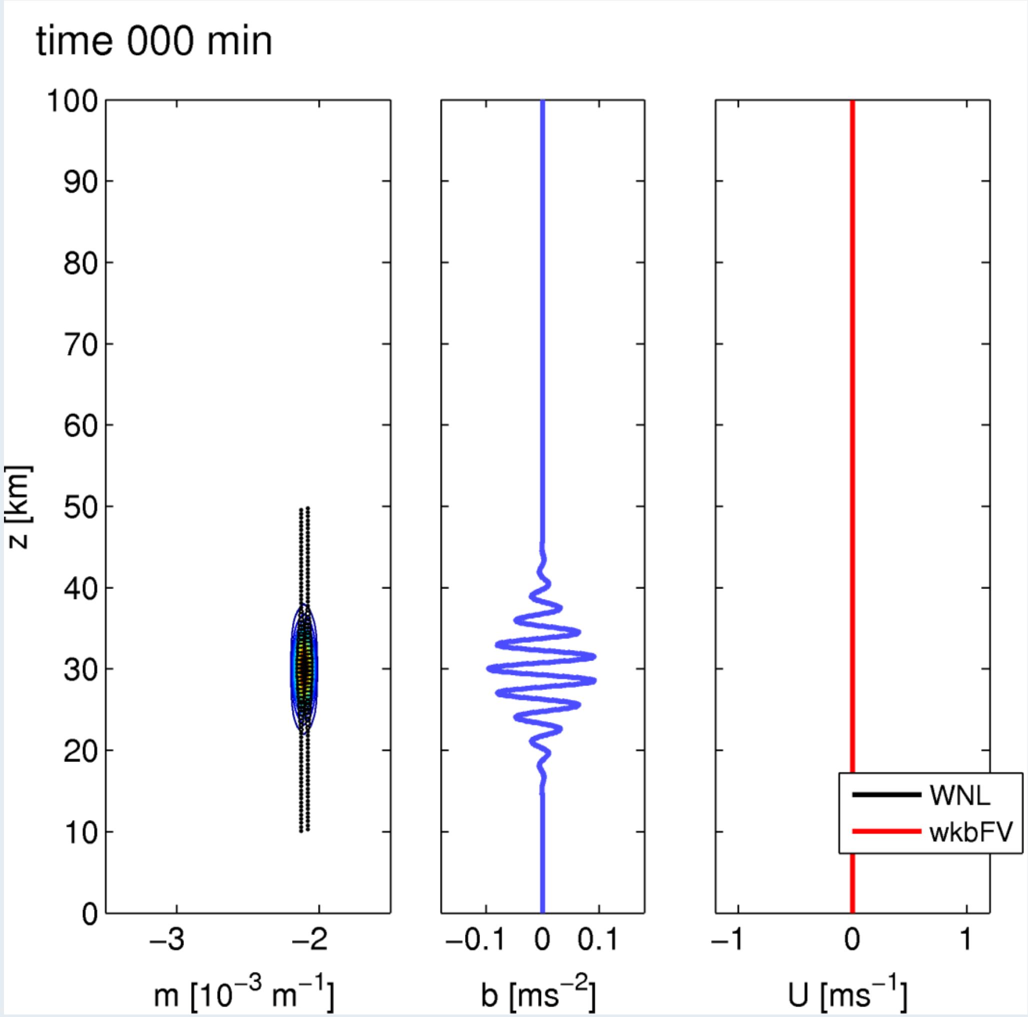
$$A_b(z) = a_0 \frac{N_0^2}{m_0}\ e^{\left[-\ \frac{(z - z_0)^2}{2\sigma^2}\right]}$$

WKB initialized with

$$\mathcal{N}(m, z, t = 0) = \begin{cases} \dfrac{A_b^2(z)}{2 N_0^2 \widehat{\omega}_0} \ \dfrac{1}{\Delta m_0} & \text{for} \quad m_0 - \dfrac{1}{2}\Delta m_0 < m < m_0 + \dfrac{1}{2}\Delta m_0 \\ 0 & \text{otherwise} \end{cases}$$

Phase-Space Wave-Action Density
(Muraschko et al 2014)

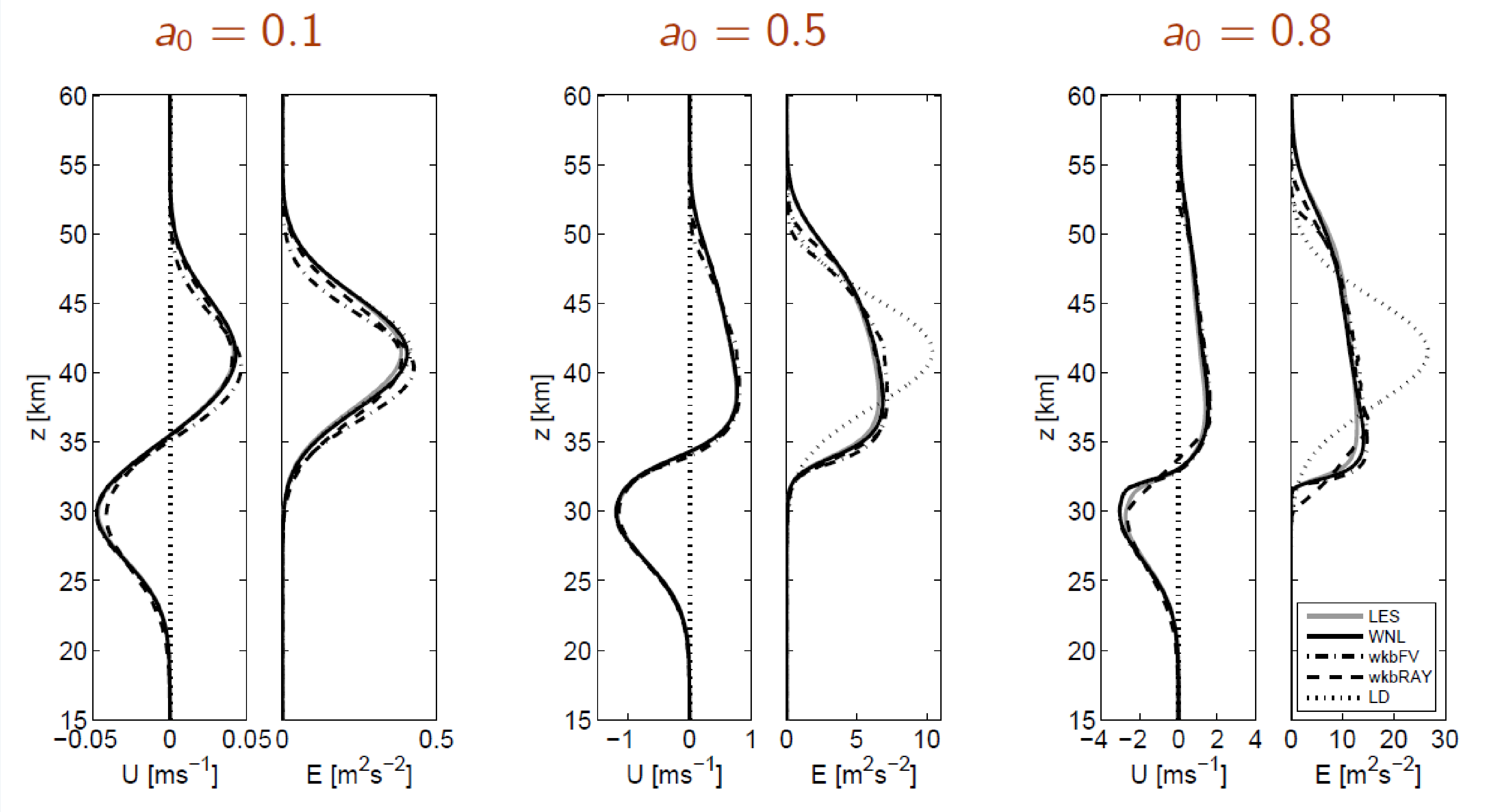
Hydrostatic wave packet



Phase-Space Wave-Action Density
(Muraschko et al 2014)

Hydrostatic wave packet:

Wave energy and induced mean flow at $t = 200$min



Phase-Space Wave-Action Density
(Muraschko et al 2014)

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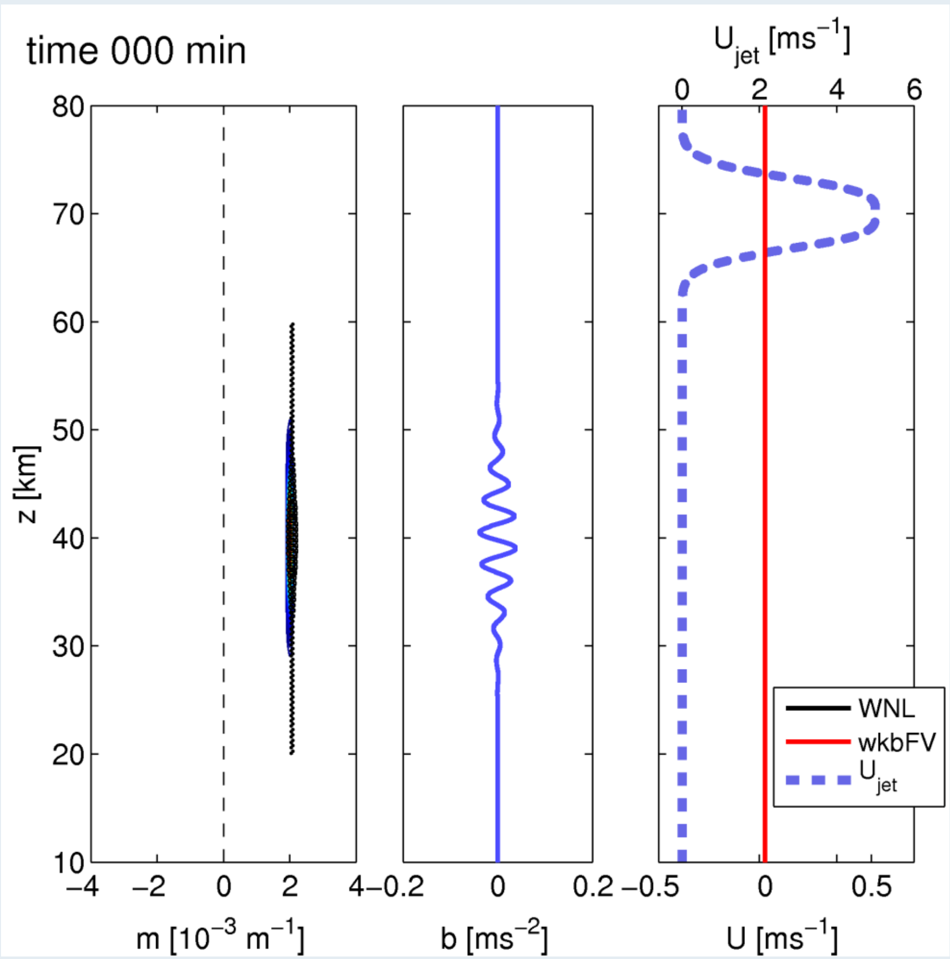
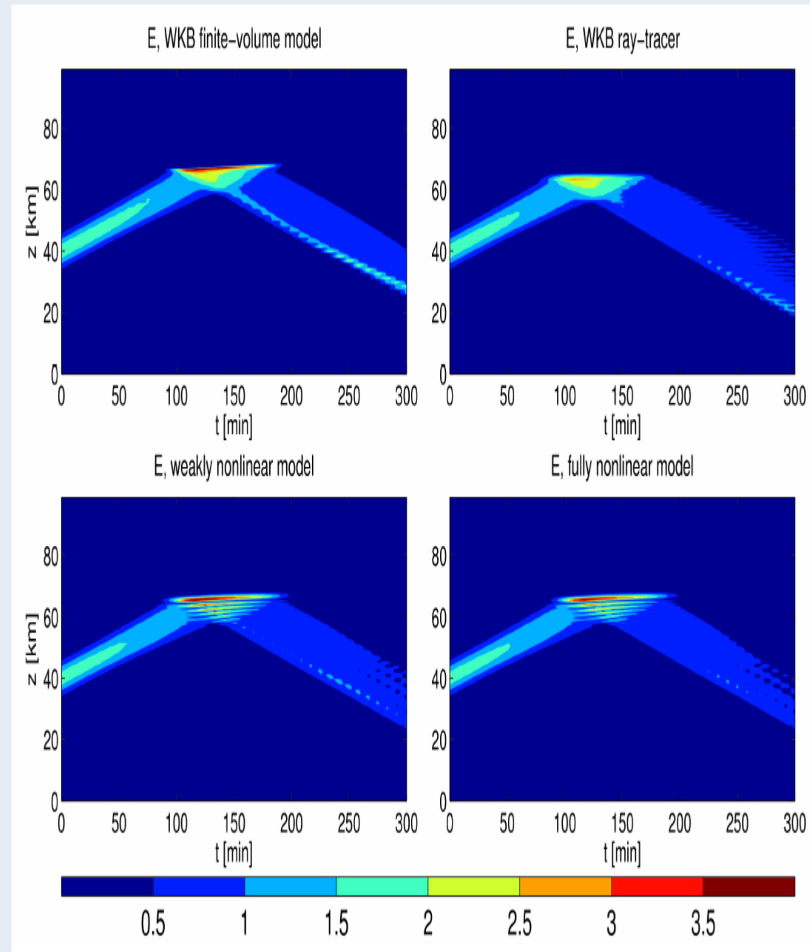
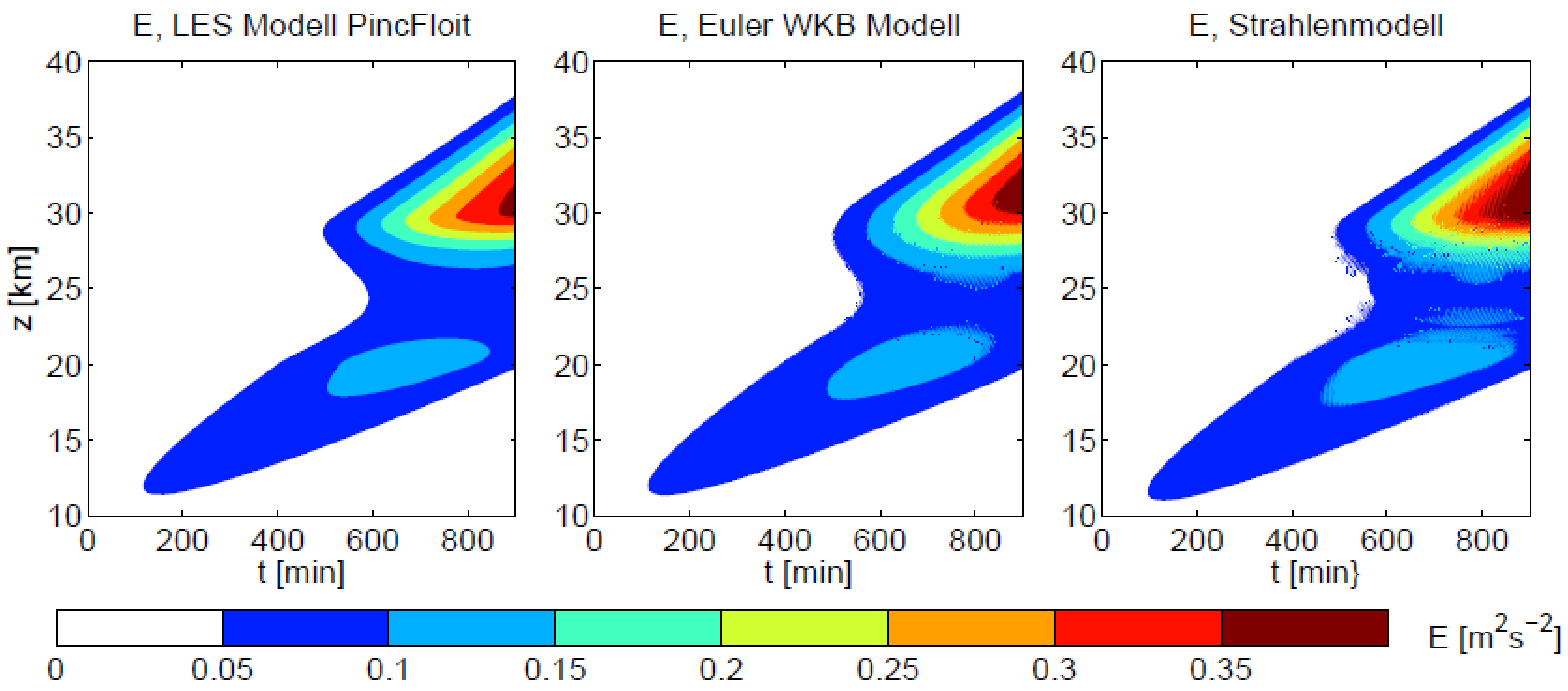
Wave packet reflected by a jet





Phase-Space Wave-Action Density
(Muraschko 2014)

Non-Boussinesq: Wave packet refracted by jet



Phase-Space Wave-Action Density
(Muraschko 2014)

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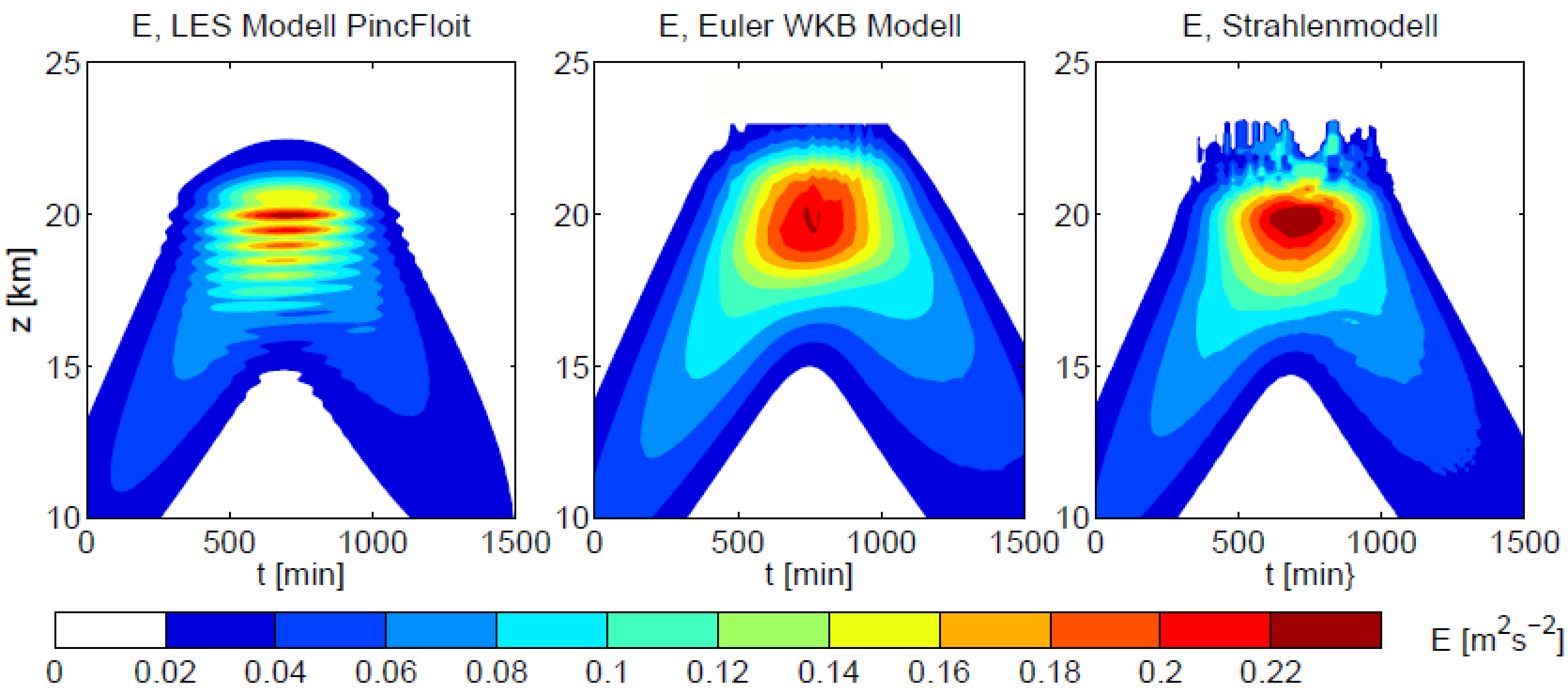
Non-Boussinesq: Wave packet reflected by a jet



Summary

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- Application of WKB to two-way interaction between propagating GW packet and induced mean flow
- Phase-space wave-action density helps avoiding numerical instabilites due to caustics
- Lagrangian approach (ray tracer) numerically efficient

Outlook: DFG research unit MS-Gwaves

12/2014-11/2017 (+ 12/2017-11/2020?)

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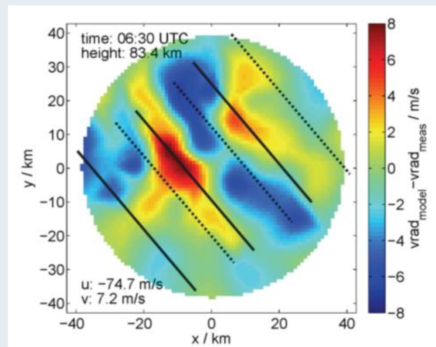
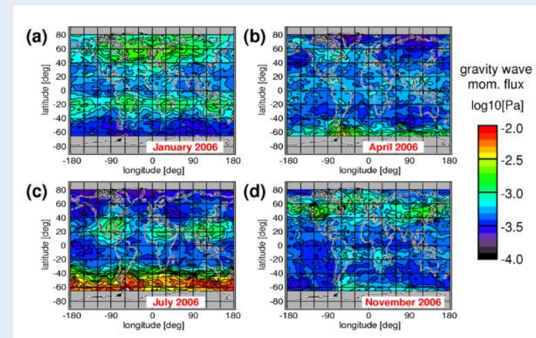
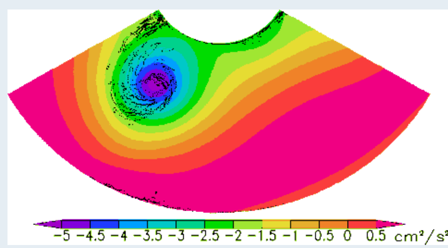
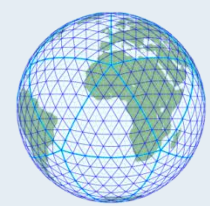
- **Investigation multi-scale dynamics of GWs in 6 projects**
- **prognostic WKB GW parameterization to be developed for NWP and climate model**
- **To be addressed:**
  - Sources
  - Propagation
  - dissipation
- **Combined effort:**
  - Theory,
  - modelling,
  - measurements,
  - laboratory experiments









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