Variational balance relations and applications

Hamburg, April 21, 2015

Marcel Oliver

Collaborators

Mahmut Çalık, David Dritschel, Georg Gottwald, Sergiy Vasylkevych

Plan

1. Variational balance relations
2. Balance models in fluid dynamics
3. PDE case study: the semilinear Klein–Gordon equation

1. Why balance models?

Balance relation as gravity wave diagnostics

- High-order balance relations?
- Mathematical properties?
- Numerical implementation?
- Data assimilation

Balance models as limiting test case for full models

- Fast rotating limits cause scale separation!

General method for certain singular perturbation problems?

- Systems with strong gyroscopic forces
- Non-relativistic limit of semilinear Klein–Gordon
- Modified equations for variational time integrators?

1.1. Why variational?

Rigid construction

- Understand conservation law structure
- Noether's theorem persists under model reduction
- For fluids: get conservation of energy and balance model PV

Flexible construction

- Variational balance relations are far from unique
- Use this freedom to get well-posedness in standard setting
- In examples: easy choice is often a good choice

1.2. Idea

Famility of Lagrangians with small parameter ε:

$$0 = \delta S = \delta \int_{t_1}^{t_2} L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon)\,\mathrm{d}t = \int_{t_1}^{t_2} \delta q_\varepsilon^T \left(\mathrm{D}_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}_{\dot{q}} L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) \right) \mathrm{d}t$$

so that

$$\mathrm{EL}_\varepsilon[q_\varepsilon] \equiv \mathrm{D}_q L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}_{\dot{q}} L_\varepsilon(q_\varepsilon, \dot{q}_\varepsilon) = 0$$

Introduce transformation $q_\varepsilon = \Phi[q]$:

$$0 = \delta S = \int_{t_1}^{t_2} \delta q^T \, \mathrm{D}\Phi[q]^* \left(\mathrm{D}_q L_\varepsilon\left(\Phi[q], \frac{\mathrm{d}}{\mathrm{d}t}\Phi[q] \right) - \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{D}_{\dot{q}} L_\varepsilon\left(\Phi[q], \frac{\mathrm{d}}{\mathrm{d}t}\Phi[q] \right) \right) \mathrm{d}t$$

So Euler–Lagrange equation reads

$$\mathrm{D}\Phi[q]^* \; \mathrm{EL}_\varepsilon[\Phi[q]] = 0$$

Now choose Φ such that

$$\mathrm{D}\Phi[q]^* \; \mathrm{EL}_\varepsilon[\Phi[q]] = \mathrm{EL}^n_{\text{slow}}[q] + O(\varepsilon^{n+1})$$

1.3. Turning the construction into a proof

From before:

$$\mathrm{D}\Phi[q]^* \ \mathrm{EL}_\varepsilon[\Phi[q]] = \mathrm{EL}^n_{\text{slow}}[q] + \varepsilon^{n+1}\,\mathrm{EL}^n_R[q]$$

Take a solution q of the slow equation:

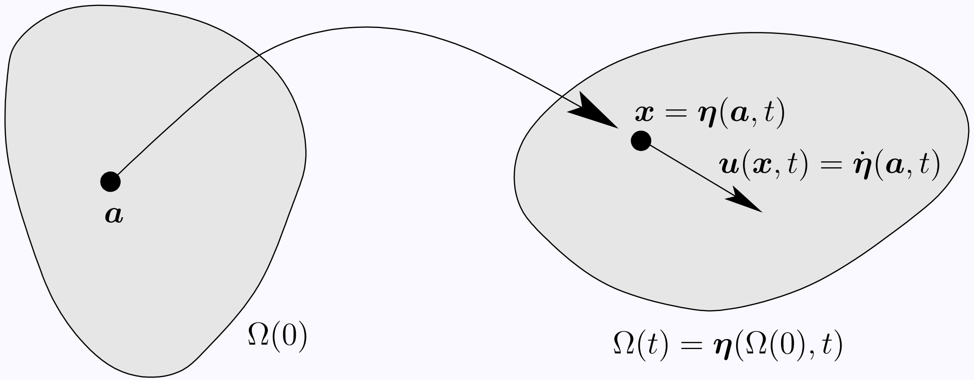
- $\mathrm{EL}^n_{\text{slow}}[q] = 0$ by definition
- Any derivative of q is $O(1)$
- Consequently, $\mathrm{EL}^n_R[q] = O(1)$
- Then $\mathrm{EL}_\varepsilon[\Phi[q]] = O(\varepsilon^{n+1})$

Conclusion:

$z \equiv \Phi[q]$ satisfies the full equation up to an $O(\varepsilon^{n+1})$ remainder.

Now use non-variational stability estimates to control the difference $q_\varepsilon - z$

2. Lagrangian fluid dynamics



For fluids, the configuration space is the group of flow maps η.

- Lagrangian vs. Eulerian fluid velocity: $\dot{\boldsymbol{\eta}} = \boldsymbol{u} \circ \boldsymbol{\eta}$
- Lagrangian vs. Eulerian variation: $\delta\boldsymbol{\eta} = \boldsymbol{w} \circ \boldsymbol{\eta}$
- Lagrangian vs. Eulerian transformation: $\boldsymbol{\eta}' = \boldsymbol{v} \circ \boldsymbol{\eta}$

Note: Affine Lagrangians (Lagrangians which are linear in the velocity) lead to kinematic Euler–Lagrange equations in Eulerian variables!

2.1. Example: Rotating shallow water

$$\varepsilon\,(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) + f\,\boldsymbol{u}^\perp + \frac{\mathrm{Bu}}{\varepsilon}\,\boldsymbol{\nabla} h = 0$$
$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0$$

- *Rossby number* $\varepsilon = U/(fL) \ll 1$
- *Burger number* $\mathrm{Bu} = gH/(f^2L^2)$

Semi-geostrophic scaling (aka. Phillips type 2 scaling/frontal geostrophic regime):

$$\mathrm{Bu} = \varepsilon$$

(Quasi-geostrophic regime is $\mathrm{Bu} = O(1)$ with $h = 1 + O(\varepsilon)$; not considered here.)

Eliassen/Hoskins: geostrophic momentum approximation

$$\varepsilon\,(\partial_t + \boldsymbol{u}_\varepsilon\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}^\perp h_\varepsilon + \boldsymbol{u}_\varepsilon^\perp + \boldsymbol{\nabla} h_\varepsilon = 0$$

- Canonical Hamiltonian system
- Advected PV in geostrophic coordinates (Hoskins, 1975)

2.2. Example ctd.: First order balance models

$$L_\varepsilon = \int h_\varepsilon \left(\boldsymbol{R} \cdot \boldsymbol{u}_\varepsilon + \tfrac{1}{2}\, \varepsilon \, |\boldsymbol{u}_\varepsilon|^2 - \tfrac{1}{2}\, h_\varepsilon \right) \mathrm{d}\boldsymbol{x}$$

where $\boldsymbol{\nabla}^\perp \cdot \boldsymbol{R} = f \equiv 1$

Expansion in ε

$$L_\varepsilon = \int h \left(\boldsymbol{R} \cdot \boldsymbol{u} - \tfrac{1}{2}\, h \right) \mathrm{d}\boldsymbol{x} + \varepsilon \int h \left(\boldsymbol{v}^\perp \cdot \boldsymbol{u} + \tfrac{1}{2}\, |\boldsymbol{u}|^2 + \tfrac{1}{2}\, h \, \boldsymbol{\nabla} \cdot \boldsymbol{v} \right) \mathrm{d}\boldsymbol{x} + O(\varepsilon^2)$$

Degeneracy condition

$$\boldsymbol{v} = \tfrac{1}{2}\, \boldsymbol{u}^\perp + \lambda \, \boldsymbol{\nabla} h$$

Salmon's (1985) L_1-model:

- Any balance model will have $\boldsymbol{u} = \boldsymbol{\nabla}^\perp h + O(\varepsilon)$, so $\lambda = \frac{1}{2}$ implies $\boldsymbol{v} = O(\varepsilon)$
- Forget the transformation!

2.3. First order model dynamics

Set $\sigma = \varepsilon(\lambda + \frac{1}{2})$. Then

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0$$
$$(q - \sigma\,\Delta)h = f$$
$$(1 - \sigma\,(h\Delta + 2\boldsymbol{\nabla} h \cdot \boldsymbol{\nabla}))\boldsymbol{u} = \boldsymbol{\nabla}^\perp\big[h - \varepsilon\,\lambda\,(2\,h\,\Delta h + |\boldsymbol{\nabla} h|^2)\big]$$

What is known:

- Derivation: Salmon (1985), O. (2006)
- Solution theory: Çalık, O., Vasylkevych (2013)
- Numerically well-behaved models, consistent initialization is difficult: Dritschel, Gottwald, O. (WIP)
- Justification: *open*

2.4. The bigger picture

Semigeostrophic equations

- Derivation: Hoskins (1975), O. (2014)
- Solution theory: Cullen, Purser, Gangbo, Feldman, … (1980s–today)
- Justification: *open*

Generalizations

- Spatially varying Coriolis parameter: O., Vasylkevych (2013)
- Stratified models: O., Vasylkevych (2013)
- Quasigeostrophic scaling, higher order models: O. (2006)

Beyond fluids

- Nonlinear oscillator in magnetic field: exponential asymptotics by Cotter and Reich (2006), variational proofs by Gottwald and O. (2014)
- The semilinear Klein–Gordon equation (to follow)
- Analysis of variational time-integrators

3. PDE case study: semilinear Klein–Gordon equation

$$\frac{\hbar^2}{2mc^2}\,\ddot{\Psi} - \frac{\hbar^2}{2m}\,\Delta\Psi + \frac{mc^2}{2}\,\Psi + f(|\Psi|^2)\Psi = 0$$

Modulated wave function

$$\psi = \Psi\,\mathrm{e}^{\frac{\mathrm{i}mc^2 t}{\hbar}}$$

Then

$$\mathrm{i}\hbar\,\dot{\psi} - \frac{\hbar^2}{2mc^2}\,\ddot{\psi} + \frac{\hbar^2}{2m}\,\Delta\psi - f(|\psi|^2)\psi = 0$$

Non-relativistic limit $c \to \infty$

- Convergence to NLS in energy space (Machihara, Nakanishi, Ozawa, Masmudi, 2000s)
- Structurally a “semigeostrophic” limit
- Can we use variational methods to derive a hierarchy of “balance models” for slow motion in the weakly relativistic regime?

3.1. Setup

Lagrangian (non-dimensionalized)

$$L(u,\dot u) = \int_{\mathbb{T}} \left(\frac{\varepsilon}{2}\,|\dot u|^2 + \frac{\mathrm{i}}{2}\,\dot u\,\overline{u} - \frac{1}{2}\,|u_x|^2 + V(u,\overline{u}) \right) \mathrm{d}x$$

Full model as first order system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \Delta/\varepsilon & \mathrm{i}/\varepsilon \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(u)/\varepsilon \end{pmatrix}$$

Notation: $g(u) = f(|u|^2)u$ where $V(u,\overline{u}) = \frac{1}{2}\,F(|u|^2)$ with $F' = f$.

Eigenoperators of linear part

$$L_\pm = \mathrm{i}\,\frac{1 \pm \sqrt{1-4\varepsilon\Delta}}{2\,\varepsilon}$$

Anatz for fast variable – remove linear slow motion *to all orders*

$$w = v - \mathrm{i}L\_u - F_{\text{slow}}^{n+1}(u)$$

3.2. Recurrence relation for slow vector field

Slow-fast splitting

$$
\begin{aligned}
\dot{u} &= \mathrm{i}L\_u + F^n_{\text{slow}}(u) + \varepsilon^{n+1}\, f_{n+1}(u) + w \\
\dot{w} &= \left(\frac{\mathrm{i}}{\varepsilon} - \mathrm{i}L\_ - DF^{n+1}_{\text{slow}}(u)\right) w + \frac{1}{\varepsilon}\left(g(u) + \mathrm{i}(1 - \varepsilon L\_)\, F^{n+1}_{\text{slow}}(u)\right) \\
&\quad - \mathrm{i}DF^{n+1}_{\text{slow}}(u)L\_u - DF^{n+1}_{\text{slow}}(u)\, F^{n+1}_{\text{slow}}(u)
\end{aligned}
$$

Construction of the slow vector field

$$
M^{-1} \equiv 1 - \varepsilon L\_ = \frac{1 + \sqrt{1 - 4\varepsilon\Delta}}{2}
$$

is positive, self-adjoint, first-order with compact inverse M. Thus,

$$
\begin{aligned}
f_0(u) &= \mathrm{i}Mg(u) \\
f_{\ell+1} &= M\left(Df_\ell(u)L\_u - \mathrm{i}\sum_{j+k=\ell} Df_j(u)f_k(u)\right)
\end{aligned}
$$

No recurrent loss of regularity! Persistence of L^2-smallness of w can be achieved to any order in ε *uniformly in the regularity class of the initial data*.

3.3. Variational asymptotics for *linear* Klein–Gordon

Quadratic action functional

$$S_\varepsilon = \frac{\varepsilon}{2}\left\langle T^2 u_\varepsilon, u_\varepsilon \right\rangle + \frac{1}{2}\left\langle T u_\varepsilon, u_\varepsilon \right\rangle + \frac{1}{2}\left\langle \Delta u_\varepsilon, u_\varepsilon \right\rangle .$$

where $\langle \cdot\,, \cdot \rangle$ is the space-time inner product and $T \equiv \mathrm{i}\,\dfrac{\partial}{\partial t}$ is formally self-adjoint.

Degeneracy condition: Can we choose $u_\varepsilon = \phi(\varepsilon T, \varepsilon\Delta)u$ such that

$$S_\varepsilon = \frac{1}{2}\left\langle (\varepsilon\, T^2 + T + \Delta)\, \phi^2(\varepsilon T, \varepsilon\Delta) u, u \right\rangle \stackrel{?}{=} \frac{1}{2}\left\langle (T + \Delta\, \theta(\varepsilon\Delta)) u, u \right\rangle$$

I.e., find *generating functions* $\phi(\xi, \eta)$ and $\theta(\eta)$, analytic near the origin, with

$$(\xi^2 + \xi + \eta)\, \phi^2(\xi, \eta) = \xi + \eta\, \theta(\eta)$$

It can be shown that there is a *unique* choice, namely

$$\theta(\eta) = \frac{1 - \sqrt{1 - 4\eta}}{2\eta} \quad \text{and} \quad \phi(\xi, \eta) = \frac{\sqrt{k(\eta)}}{\sqrt{1 + \xi k(\eta)}} \quad \text{with} \quad k(\eta) = \frac{2}{1 + \sqrt{1 - 4\eta}}$$

3.4. Expansion of the linear transformation

- When plugging the linear transformation into the potential, we need to expand
- Can we do this without losing derivatives?

Naive expansion

Let K be the compact operator with symbol k. Then

$$\phi(\varepsilon T, \varepsilon\Delta) = \frac{\sqrt{K}}{\sqrt{1+\varepsilon TK}} = \sqrt{K} \sum_{j=0}^{\infty} \binom{-\frac{1}{2}}{j} \varepsilon^j (TK)^j$$

On solutions of the slow equation, TK is a zero order operator,
but its operator norm is $O(\varepsilon^{-1})$ unless we lose derivatives

Better expansion – use lower order “balance”

$$\phi(\varepsilon T, \varepsilon\Delta) = \frac{\sqrt{K}}{\sqrt{1+\varepsilon TK}} = \frac{\sqrt{M}}{\sqrt{1+\varepsilon (T+L_-)M}} \qquad \text{with} \qquad M = \frac{1}{\sqrt{1-4\varepsilon\Delta}}$$

On solutions of the slow equation, $(T+L_-)M$ is of zero order uniformly!

3.5. Shadowing theorem

Let u denote a solution of the slow Euler–Lagrange equation $u(0) \in H^2$. Let u_ε solve the full Euler–Lagrange equation consistently initialized via

$$u_\varepsilon(0) = \Phi_n[u]\Big|_{t=0}$$
$$\dot{u}_\varepsilon(0) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi_n[u]\Big|_{t=0}$$

Then for every fixed $T > 0$ there exist $\varepsilon_0 > 0$ and $c = c(u(0), T)$ such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$\sup_{t \in [0,T]} \|u_\varepsilon(t) - \Phi_n[u(t)]\|_{L^2} \leq c\,\varepsilon^{n+1}$$

Proof

Note that all operators are bounded – proceed as in finite dimensions.

Conclusion

Hamiltonian PDE asymptotics without "loss of derivatives"